

Methods of chaos detection

Haris Skokos

**Department of Mathematics and Applied Mathematics
University of Cape Town
Cape Town, South Africa**

E-mail: haris.skokos@uct.ac.za

URL: http://math_research.uct.ac.za/~hskokos/

Outline

- **Dynamical Systems**
 - ✓ **Hamiltonian models – Variational equations**
 - ✓ **Symplectic maps – Tangent map**
- **Brief description of chaos detection methods**
- **Chaos Indicators**
 - ✓ **Lyapunov exponents**
 - ✓ **Smaller ALignment Index – SALI**
 - **Definition**
 - **Behavior for chaotic and regular motion**
 - **Applications**
 - ✓ **Generalized ALignment Index – GALI**
 - **Definition - Relation to SALI**
 - **Behavior for chaotic and regular motion**
 - **Application to time-dependent models**

Autonomous Hamiltonian systems

Consider an **N degree of freedom** autonomous Hamiltonian system having a Hamiltonian function of the form:

$$H(\overbrace{q_1, q_2, \dots, q_N}^{\text{positions}}, \overbrace{p_1, p_2, \dots, p_N}^{\text{momenta}})$$

The time evolution of an orbit (trajectory) with initial condition

$$P(0) = (q_1(0), q_2(0), \dots, q_N(0), p_1(0), p_2(0), \dots, p_N(0))$$

is governed by the **Hamilton's equations of motion**

$$\frac{dp_i}{dt} = - \frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

Variational Equations

We use the notation $\mathbf{x} = (q_1, q_2, \dots, q_N, p_1, p_2, \dots, p_N)^T$. The **deviation vector** from a given orbit is denoted by

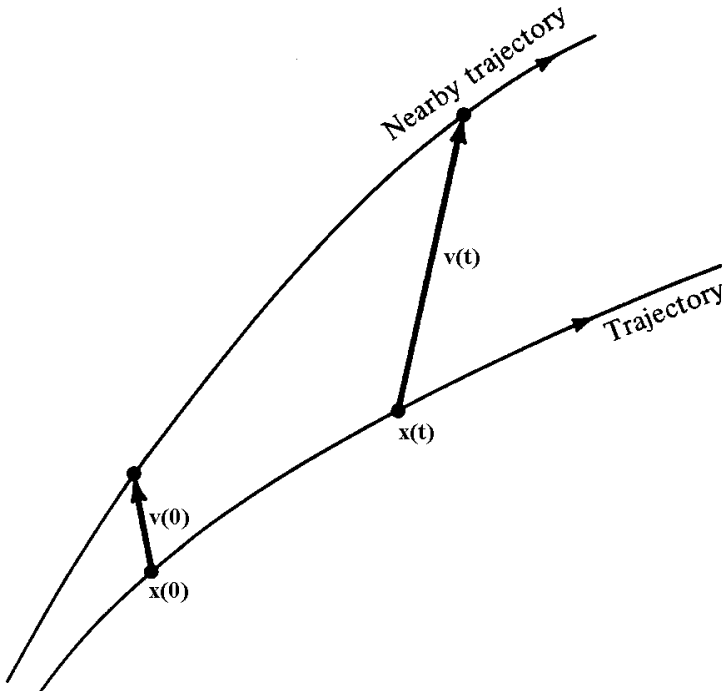
$$\mathbf{v} = (\delta x_1, \delta x_2, \dots, \delta x_n)^T, \text{ with } n=2N$$

The time evolution of \mathbf{v} is given by the so-called **variational equations**:

$$\frac{d\mathbf{v}}{dt} = -\mathbf{J} \cdot \mathbf{P} \cdot \mathbf{v}$$

where

$$\mathbf{J} = \begin{pmatrix} \mathbf{0}_N & -\mathbf{I}_N \\ \mathbf{I}_N & \mathbf{0}_N \end{pmatrix}, \quad P_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j} \quad i, j = 1, 2, \dots, n$$



Symplectic Maps

Consider an **2N-dimensional symplectic map T**. In this case we have **discrete time**.

The evolution of an **orbit** with initial condition

$$P(0)=(x_1(0), x_2(0), \dots, x_{2N}(0))$$

is governed by the **equations of map T**

$$P(i+1)=T P(i) \text{ , } i=0,1,2,\dots$$

The evolution of an initial **deviation vector**

$$v(0) = (\delta x_1(0), \delta x_2(0), \dots, \delta x_{2N}(0))$$

is given by the corresponding **tangent map**

$$v(i+1) = \left. \frac{\partial T}{\partial P} \right|_i \cdot v(i) \text{ , } i = 0, 1, 2, \dots$$

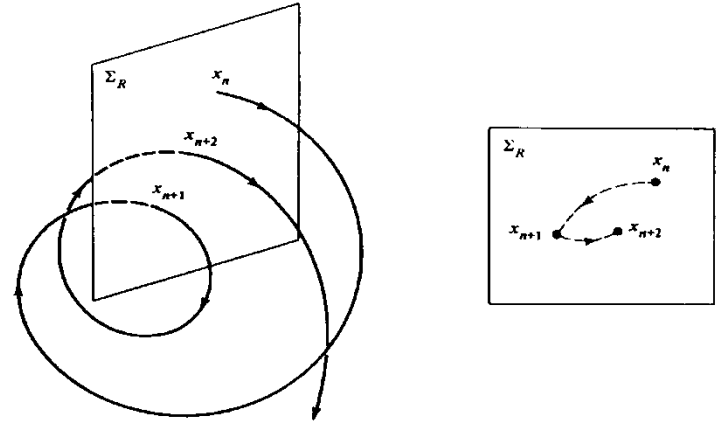
Chaos detection techniques

- **Based on the visualization of orbits**
 - ✓ **Poincaré Surface of Section (PSS)**
 - ✓ **the color and rotation (CR) method**
 - ✓ **the 3D phase space slices (3PSS) technique**

Poincaré Surface of Section (PSS)

We can constrain the study of an $N+1$ degree of freedom Hamiltonian system to a **2N-dimensional subspace of the general phase space**.

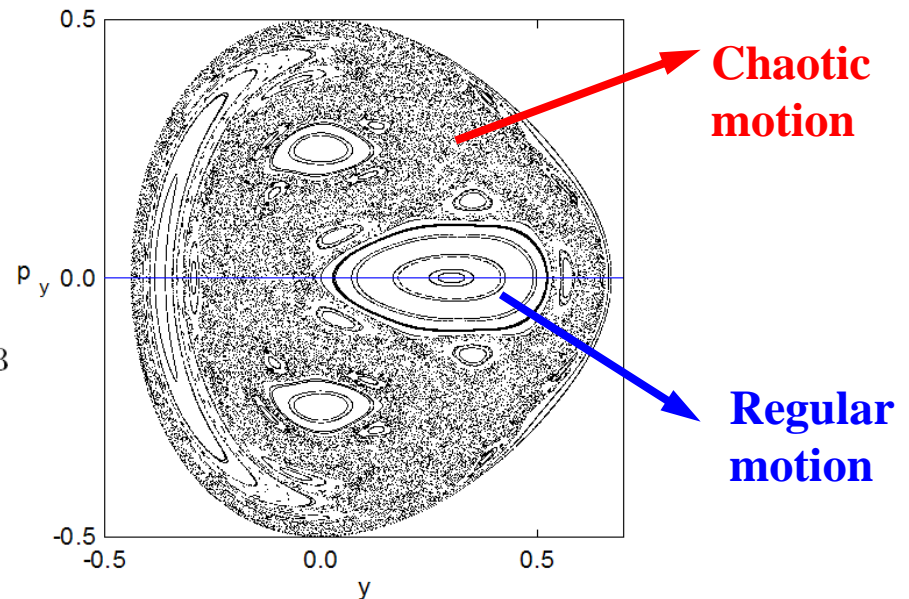
In this sense an $N+1$ degree of freedom Hamiltonian system corresponds to a **2N-dimensional symplectic map**.



Lieberman & Lichtenberg, 1992, *Regular and Chaotic Dynamics*, Springer.

The 2D Hénon-Heiles system:

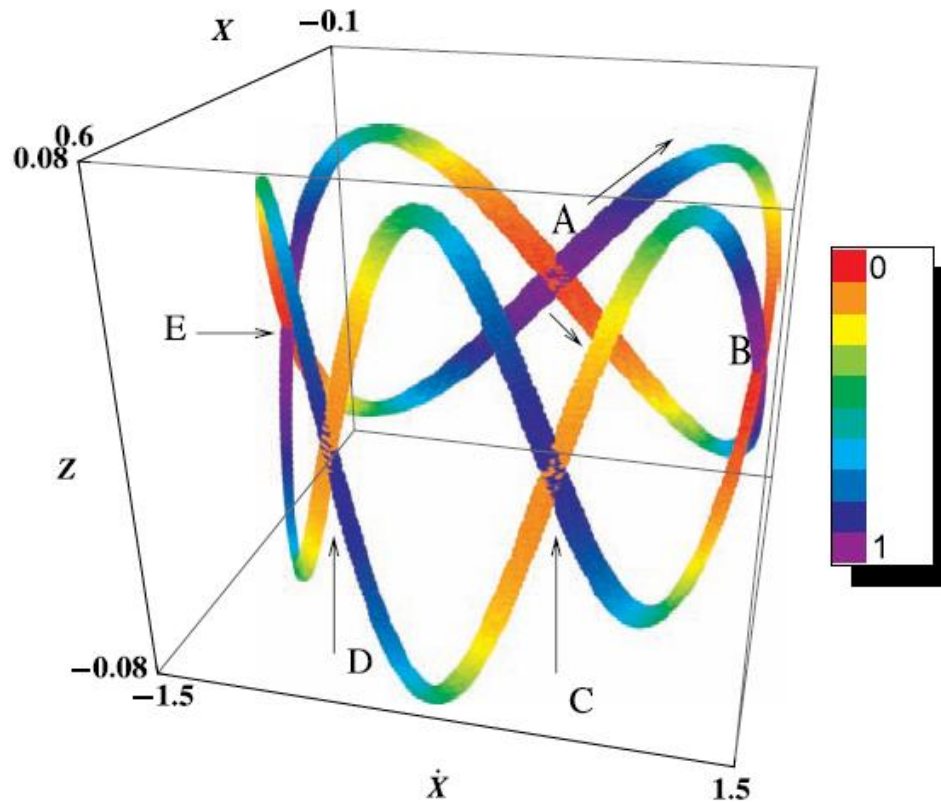
$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3$$



The color and rotation (CR) method

For 3 degree of freedom Hamiltonian systems and 4 dimensional symplectic maps:

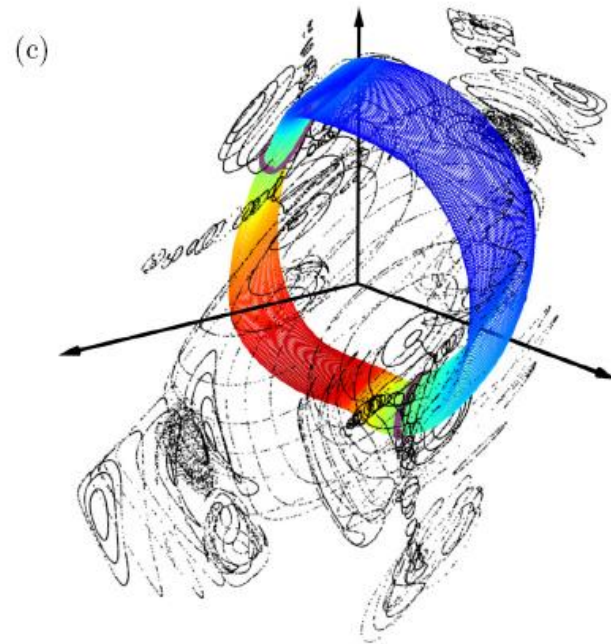
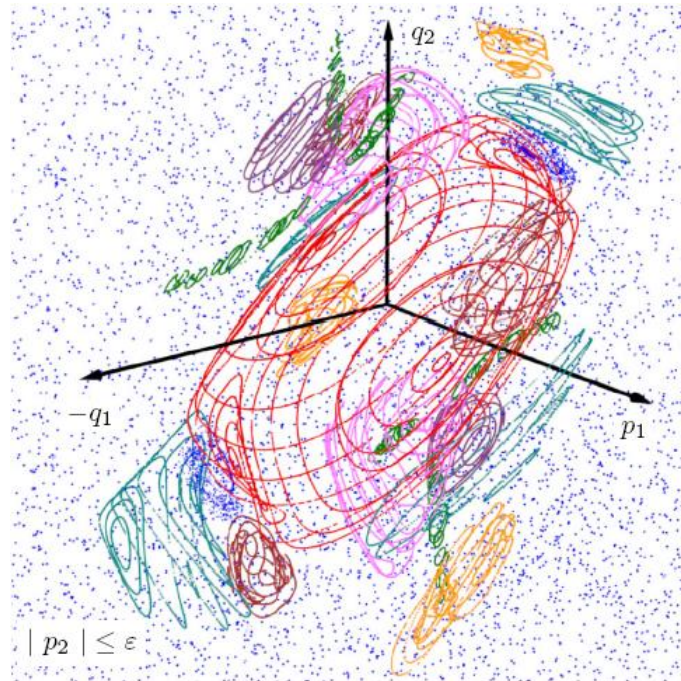
We consider the 3D projection of the PSS and use color to indicate the 4th dimension.



The 3D phase space slices (3PSS) technique

For 3 degree of freedom Hamiltonian systems and 4 dimensional symplectic maps:

We consider thin 3D phase space slices of the 4D phase space (e.g. $|p_2| \leq \varepsilon$) and present intersections of orbits with these slices.



Chaos detection techniques

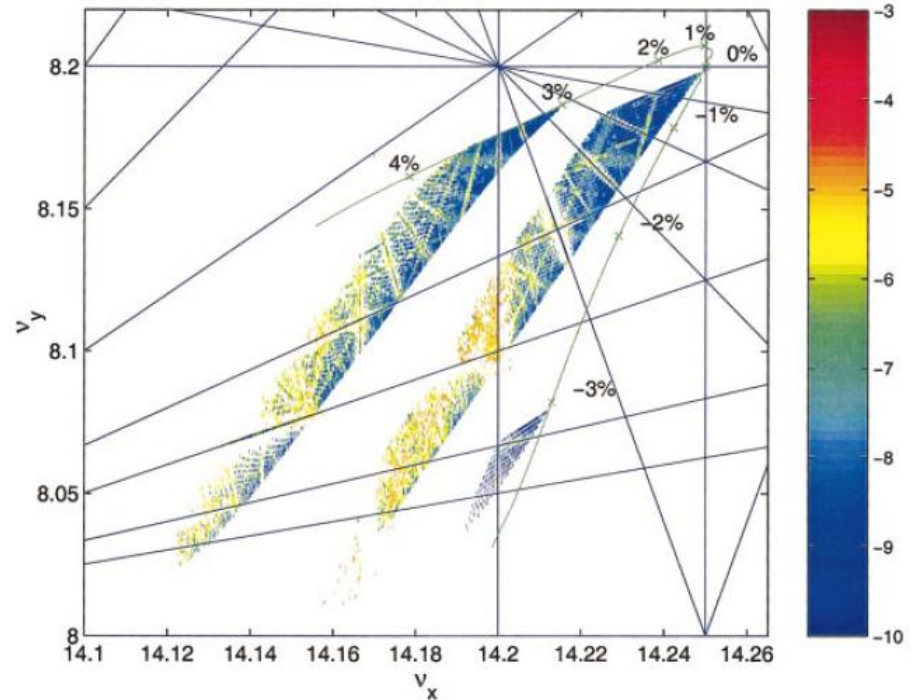
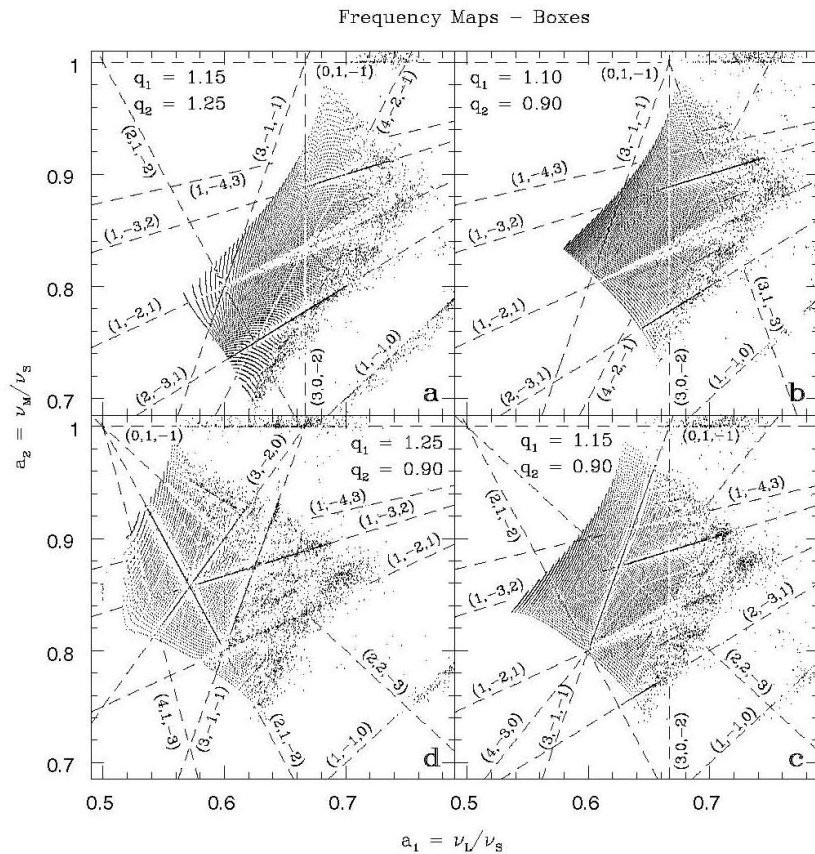
- **Based on the visualization of orbits**
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- **Based on the numerical analysis of orbits**
 - ✓ **Frequency Map Analysis**
 - ✓ **0-1 test**

Frequency Map Analysis

Create **Frequency Maps** by computing the fundamental frequencies of orbits.

Regular motion: The computed frequencies do not vary in time

Chaotic motion: The computed frequencies vary in time



Steier C et al. 2002 Phys. Rev. E 65 056506

Chaos detection techniques

- **Based on the visualization of orbits**
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- **Based on the numerical analysis of orbits**
 - ✓ **Frequency Map Analysis**
 - ✓ **0-1 test**
- **Chaos indicators based on the evolution of deviation vectors from a given orbit**
 - ✓ **Maximum Lyapunov Exponent**
 - ✓ **Fast Lyapunov Indicator (FLI) and Orthogonal Fast Lyapunov Indicators (OFLI and OFLI2)**
 - ✓ **Mean Exponential Growth Factor of Nearby Orbits (MEGNO)**
 - ✓ **Relative Lyapunov Indicator (RLI)**
 - ✓ **Smaller ALignment Index – SALI**
 - ✓ **Generalized ALignment Index – GALI**

Maximum Lyapunov Exponent

Roughly speaking, the Lyapunov exponents of a given orbit characterize the **mean exponential rate of divergence** of trajectories surrounding it.

Consider an orbit in the $2N$ -dimensional phase space with **initial condition** $\mathbf{x}(0)$ and an **initial deviation vector from it** $\mathbf{v}(0)$. Then the mean exponential rate of divergence is:

$$\text{mLCE} = \sigma_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\|\vec{v}(t)\|}{\|\vec{v}(0)\|}$$

$\sigma_1 = 0 \rightarrow$ Regular motion

$\sigma_1 \neq 0 \rightarrow$ Chaotic motion

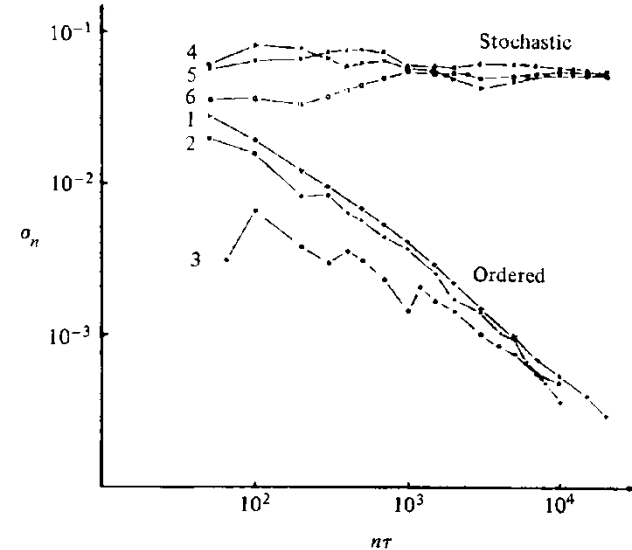


Figure 5.7. Behavior of σ_n at the intermediate energy $E = 0.125$ for initial points taken in the ordered (curves 1–3) or stochastic (curves 4–6) regions (after Benettin *et al.*, 1976).

If we start with more than one linearly independent deviation vectors they will **align to the direction defined by the largest Lyapunov exponent** for chaotic orbits.

**The
Smaller ALignment Index
(SALI)
method**

Definition of the SALI

We follow the evolution in time of two different initial deviation vectors ($\mathbf{v}_1(0)$, $\mathbf{v}_2(0)$), and define the SALI (**Ch.S. 2001, J. Phys. A**) as:

$$\text{S A L I}(t) = \min \left\{ \left\| \hat{\mathbf{v}}_1(t) + \hat{\mathbf{v}}_2(t) \right\|, \left\| \hat{\mathbf{v}}_1(t) - \hat{\mathbf{v}}_2(t) \right\| \right\}$$

where

$$\hat{\mathbf{v}}_1(t) = \frac{\mathbf{v}_1(t)}{\|\mathbf{v}_1(t)\|}$$

When the two vectors become **collinear**

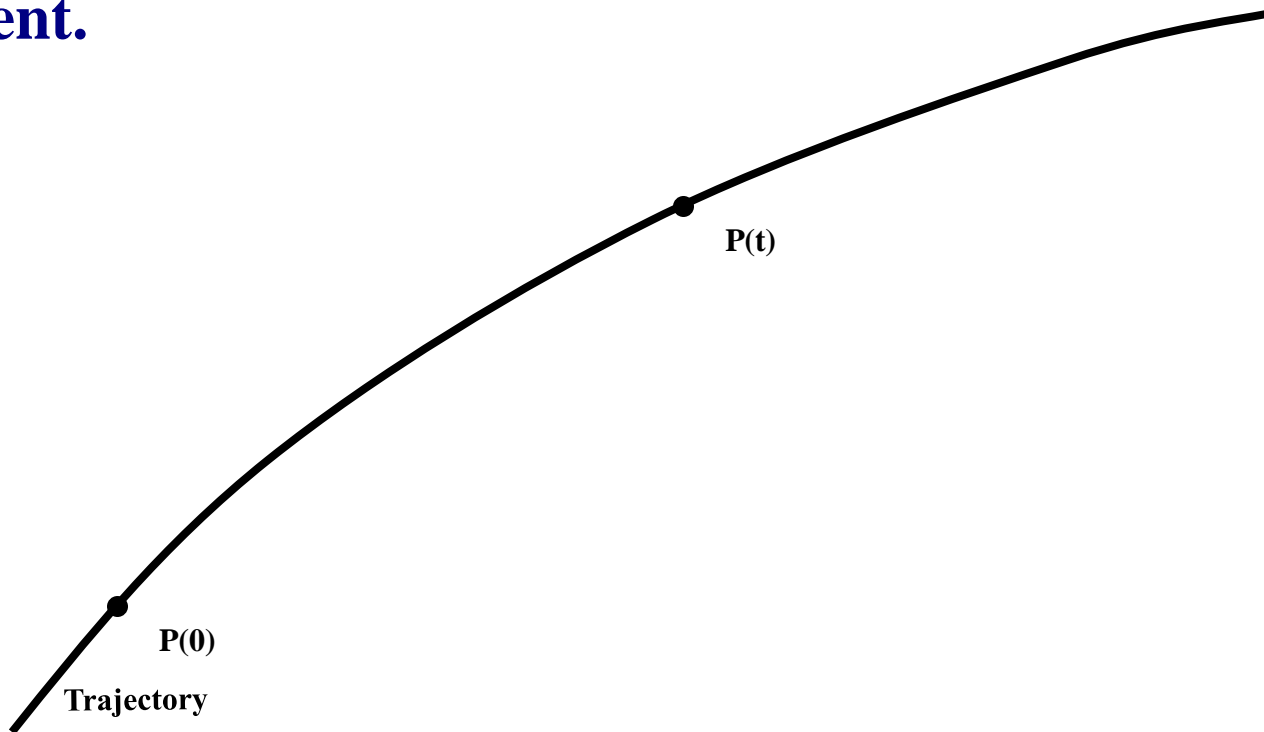
$$\text{SALI}(t) \rightarrow 0$$

Behavior of the SALI for chaotic motion

For chaotic orbits the two initially different deviation vectors tend to coincide with the direction defined by the maximum Lyapunov exponent.

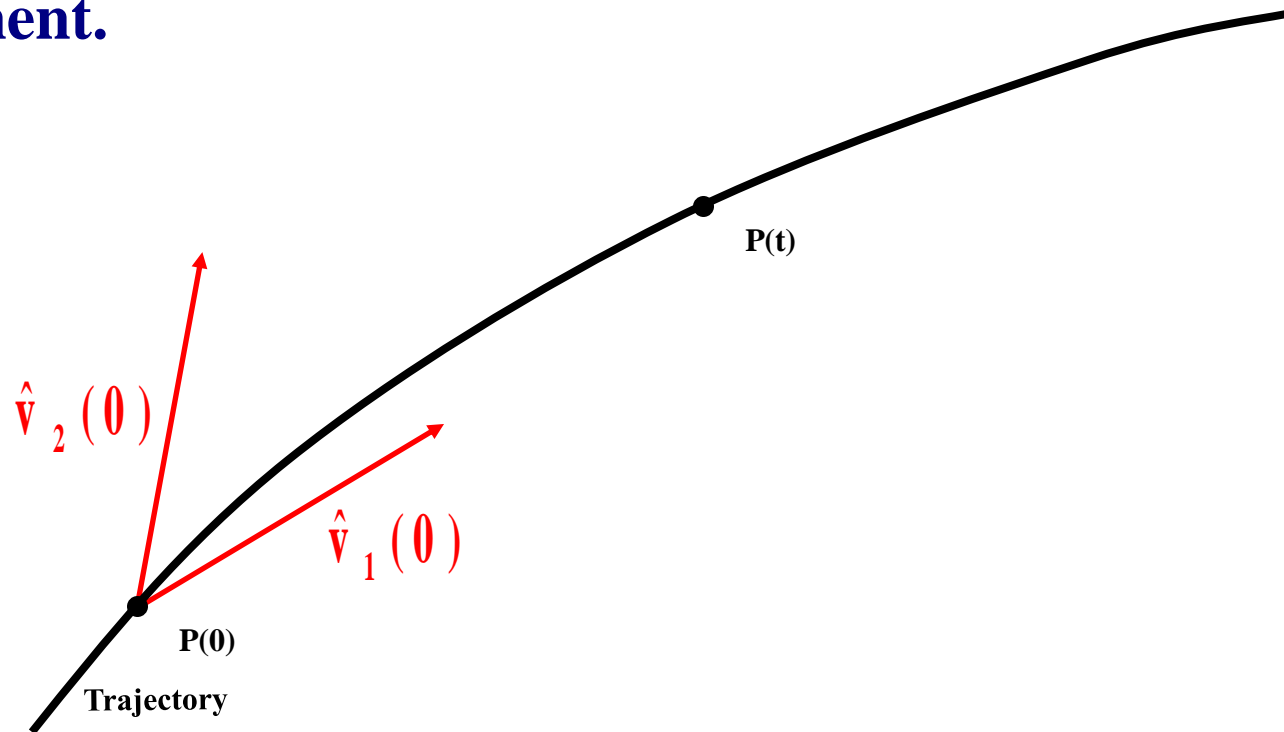
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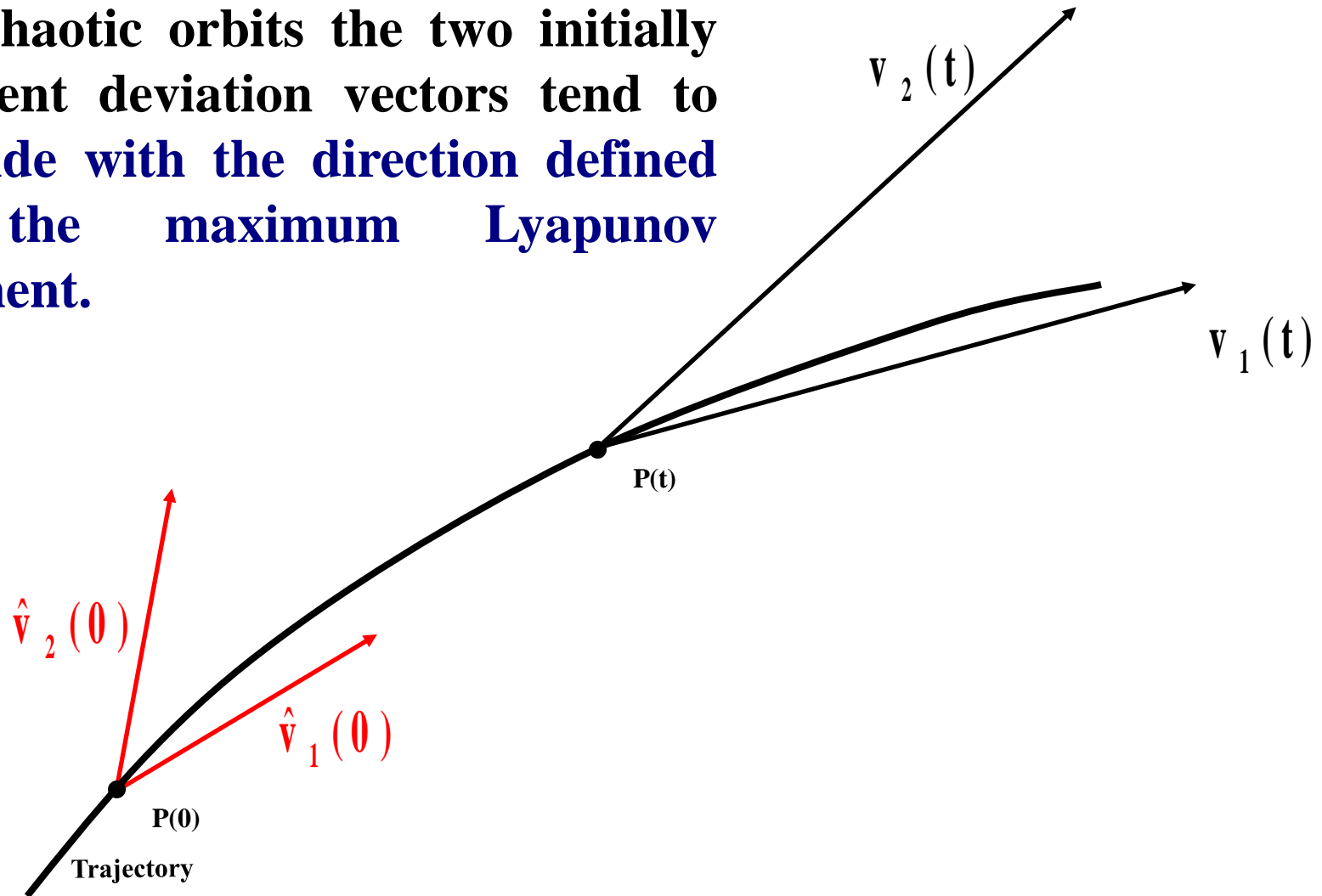
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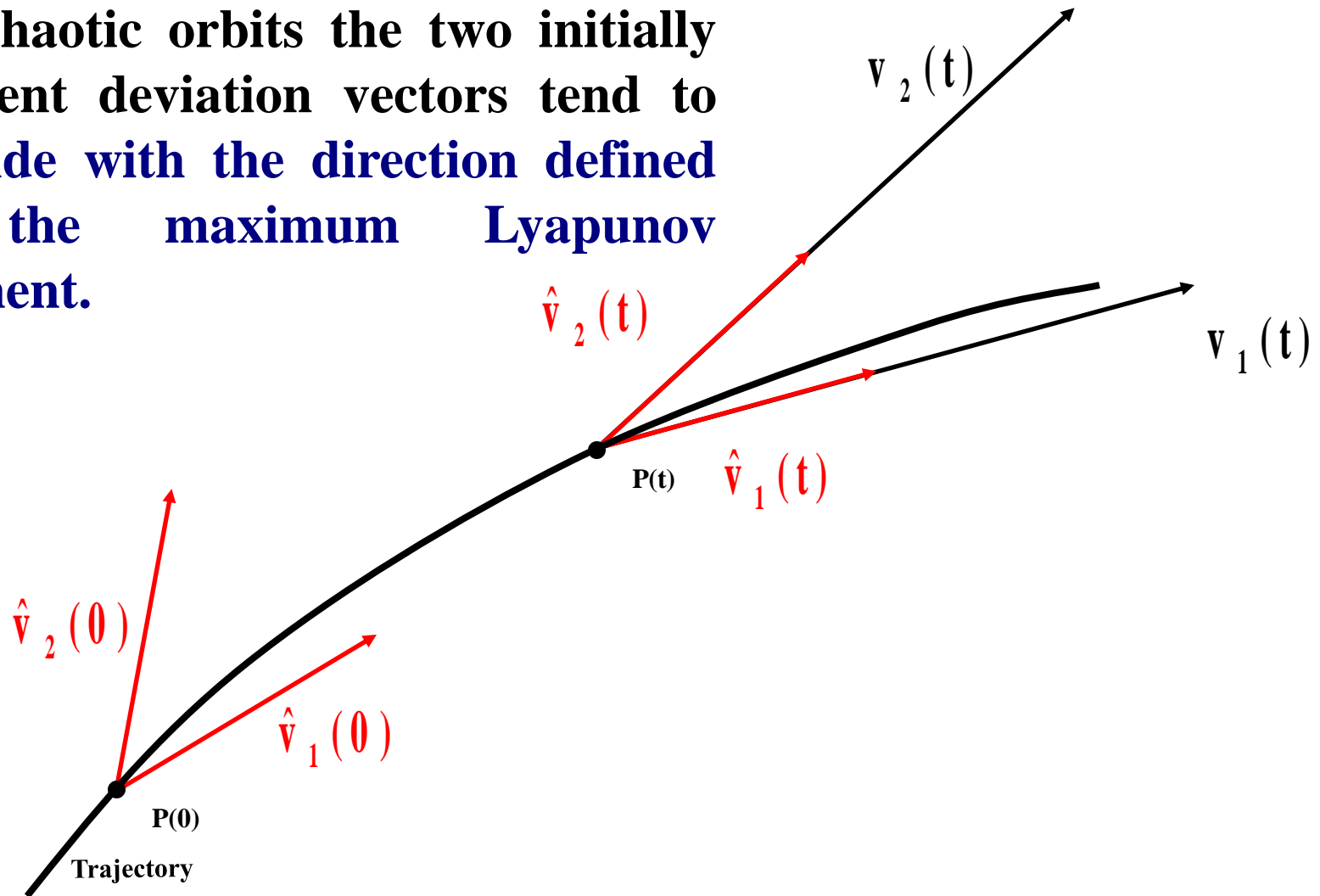
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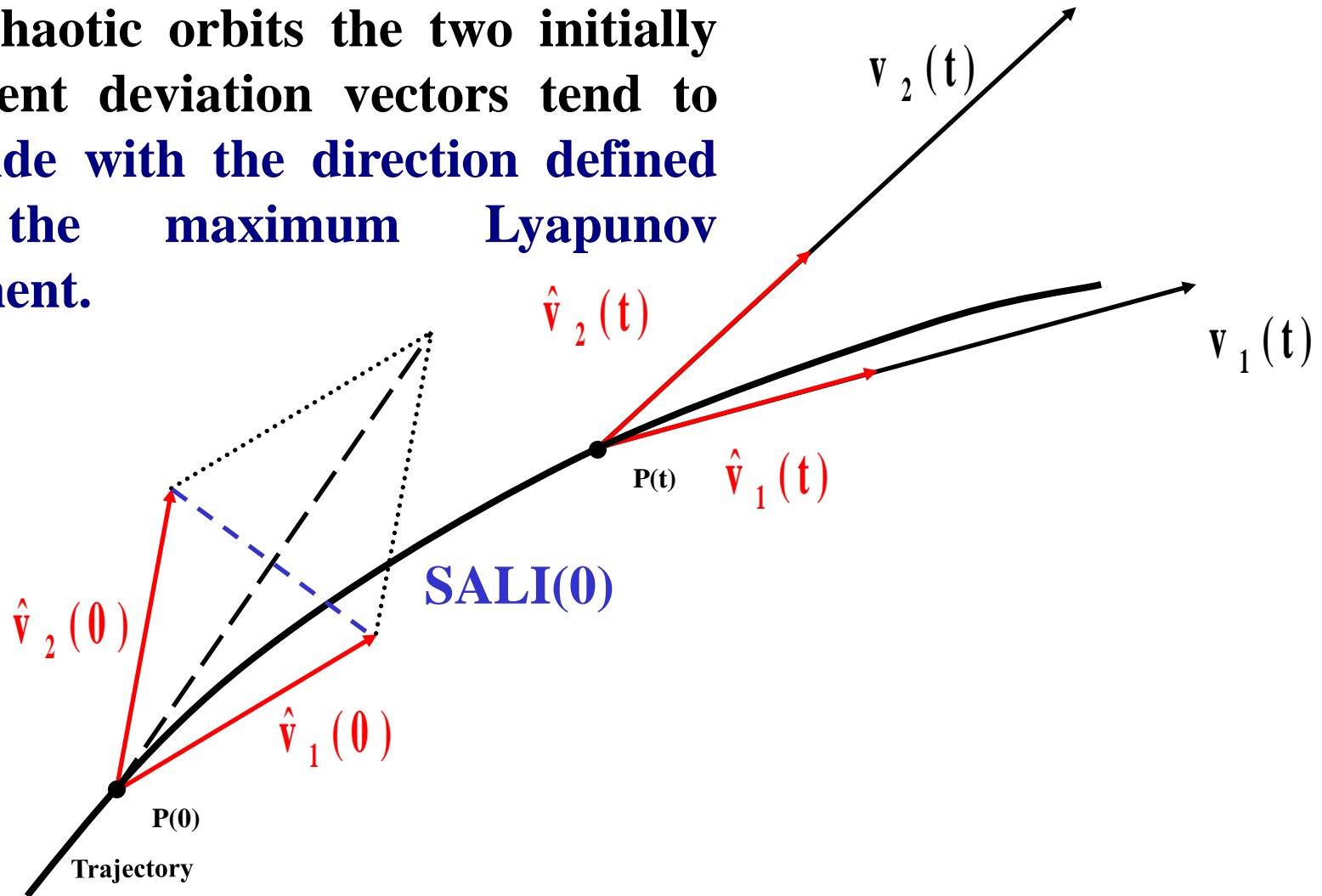
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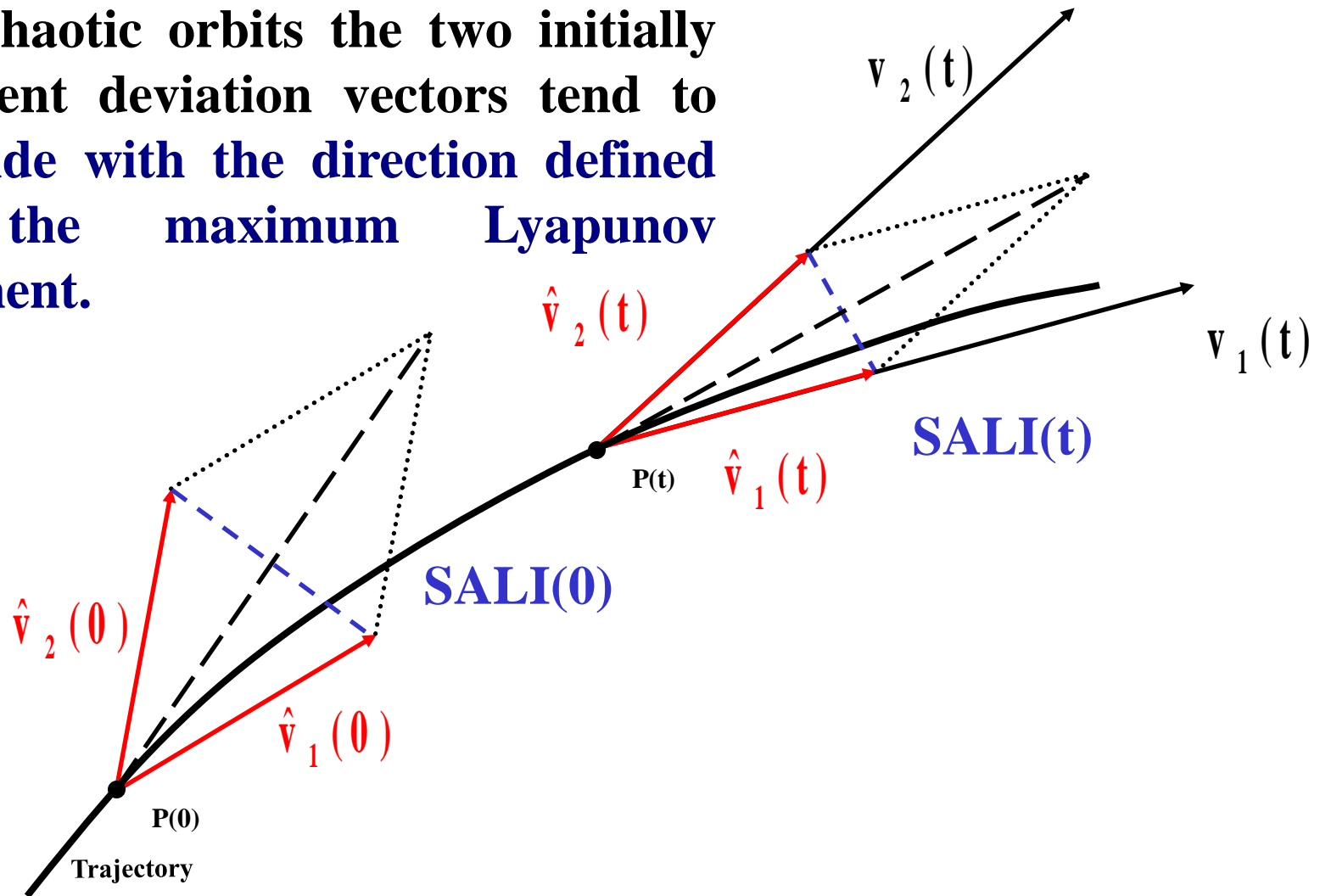
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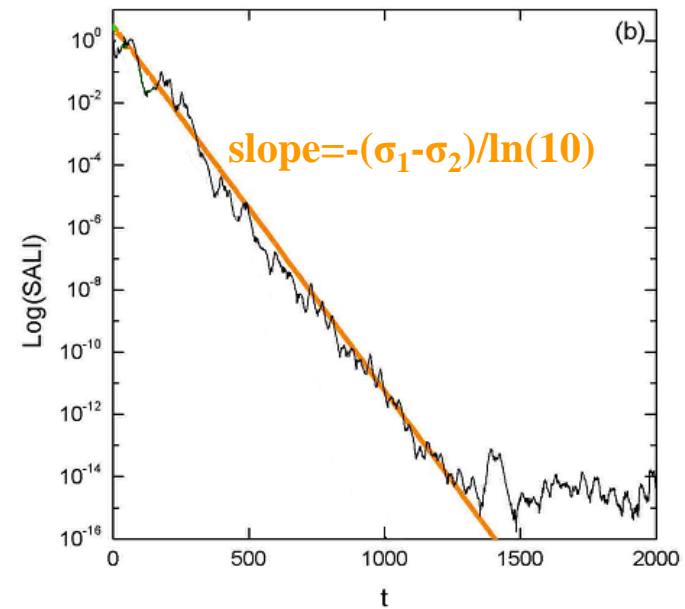
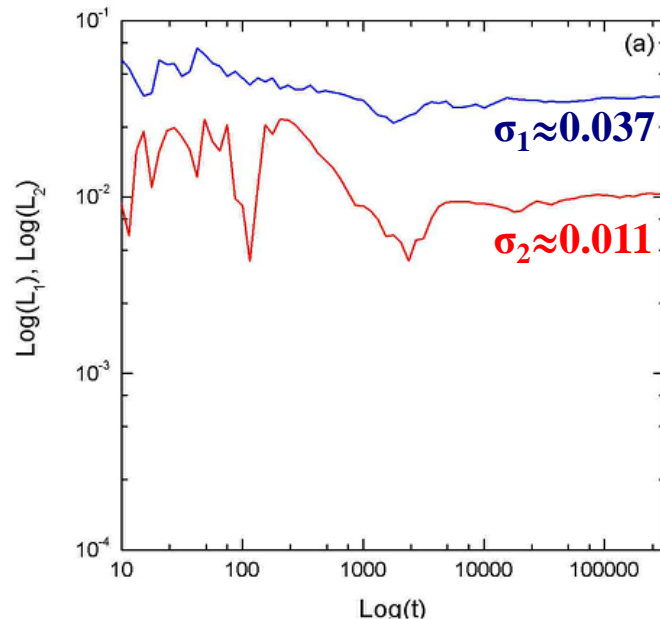


Behavior of the SALI for chaotic motion

We test the validity of the approximation $\text{SALI} \propto e^{-(\sigma_1 - \sigma_2)t}$ (Ch.S., Antonopoulos, Bountis, Vrahatis, 2004, J. Phys. A) for a chaotic orbit of the 3D Hamiltonian

$$H = \sum_{i=1}^3 \frac{\omega_i}{2} (q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3$$

with $\omega_1=1$, $\omega_2=1.4142$, $\omega_3=1.7321$, $H=0.09$

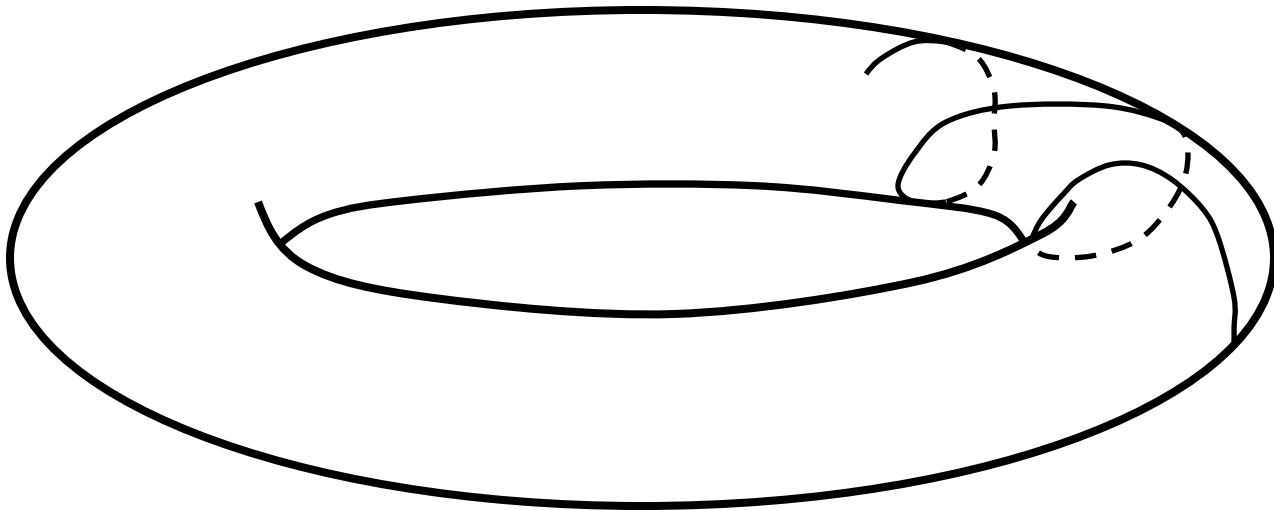


Behavior of the SALI for **regular motion**

Regular motion occurs on a torus and two different initial deviation vectors **become tangent to the torus, generally having different directions.**

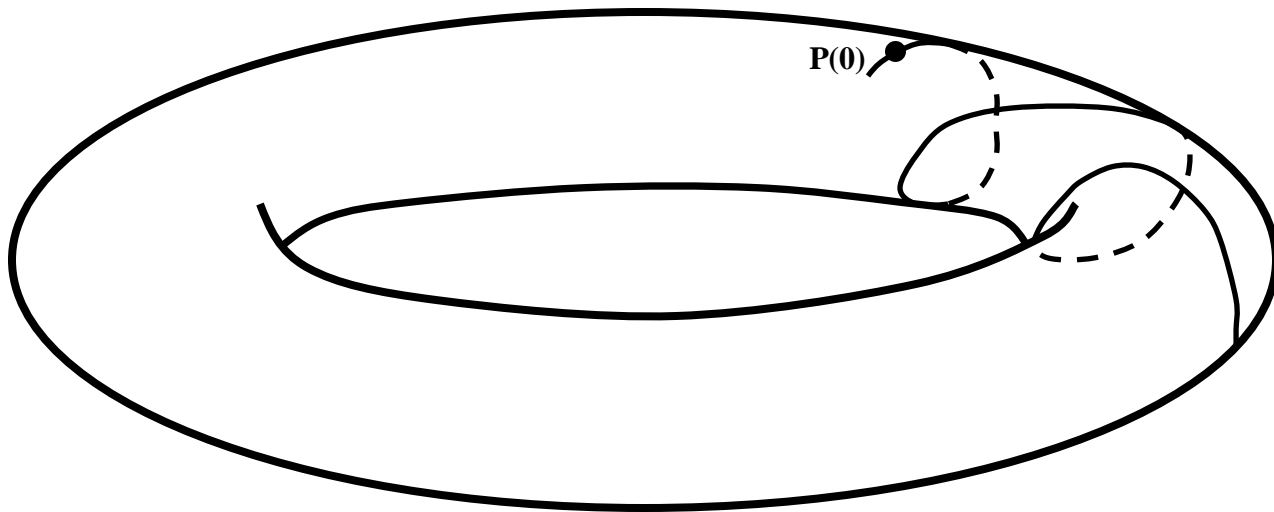
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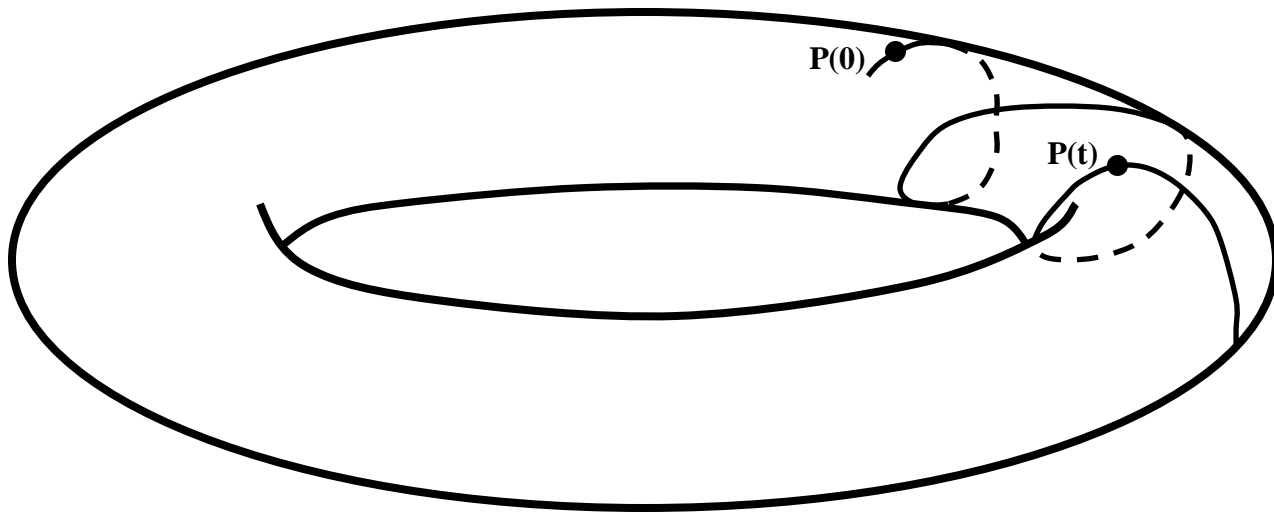
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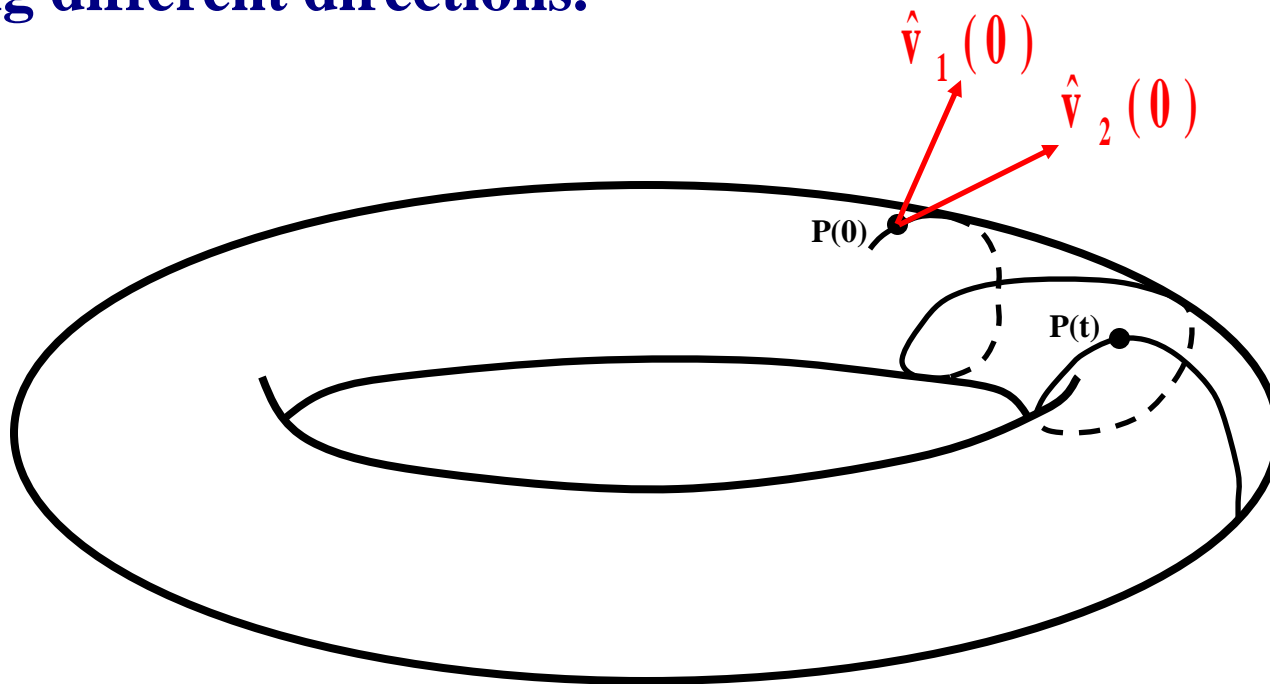
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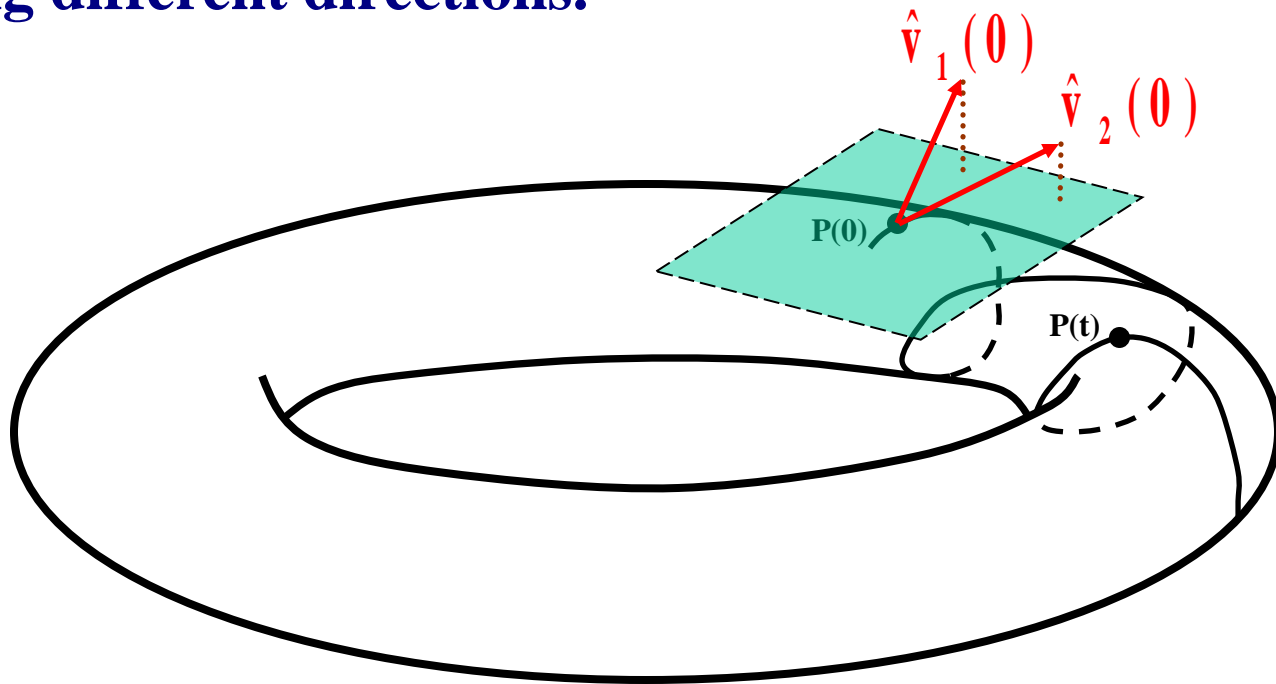
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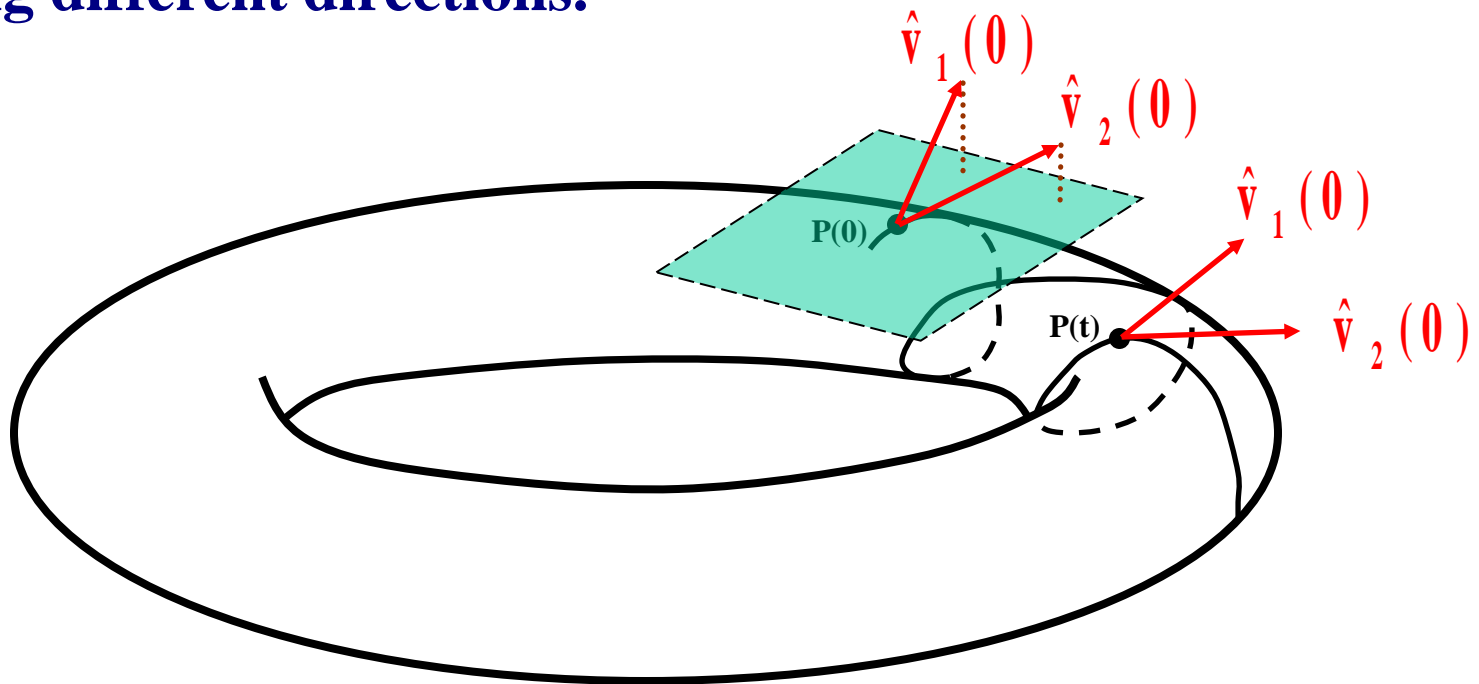
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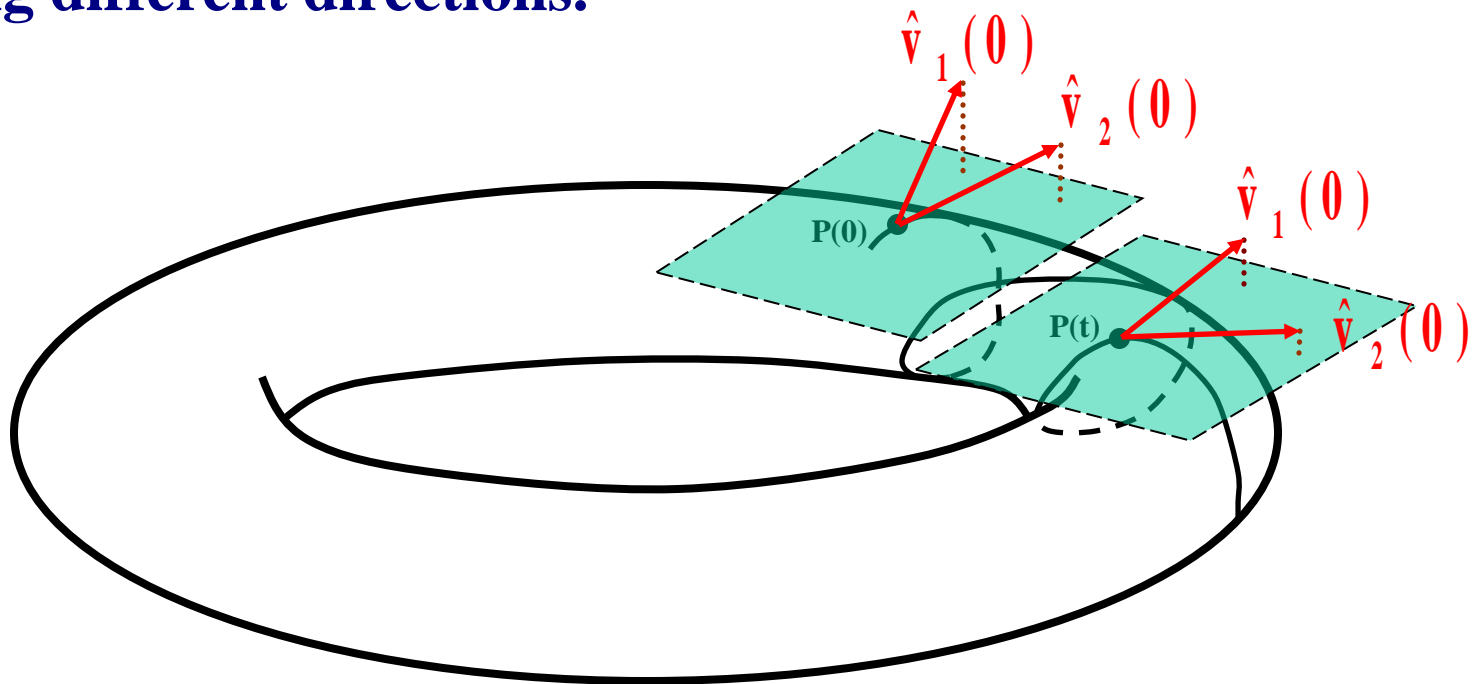
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Behavior of the SALI for **regular motion**

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Applications – Hénon-Heiles system

As an example, we consider the 2D Hénon-Heiles system:

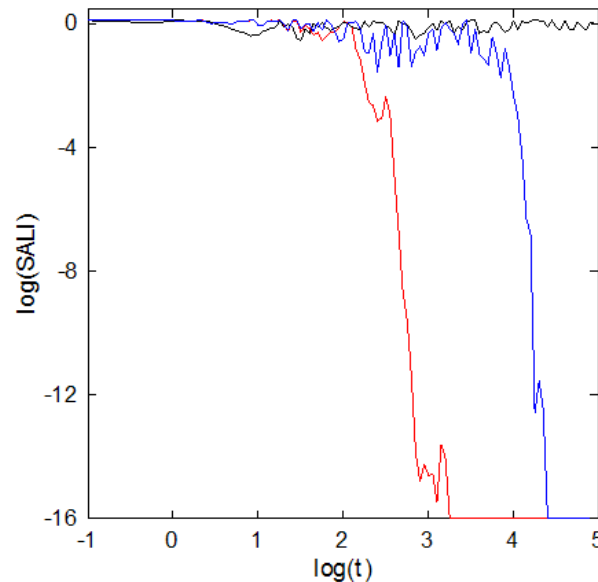
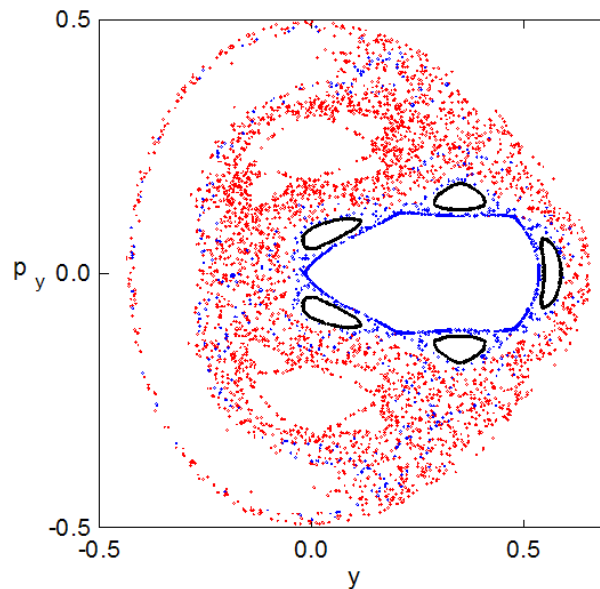
$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3$$

For $E=1/8$ we consider the orbits with initial conditions:

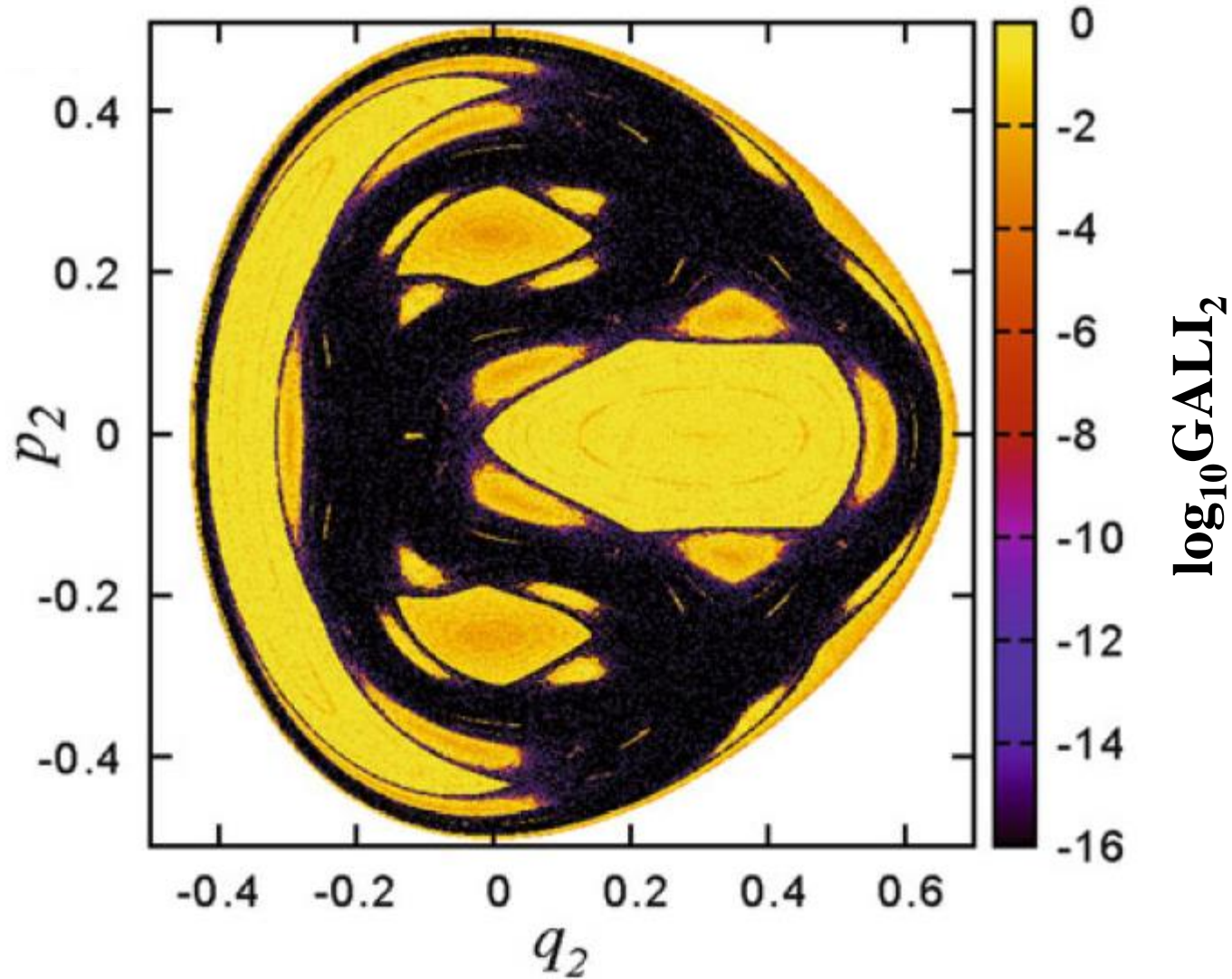
Regular orbit, $x=0$, $y=0.55$, $p_x=0.2417$, $p_y=0$

Chaotic orbit, $x=0$, $y=-0.016$, $p_x=0.49974$, $p_y=0$

Chaotic orbit, $x=0$, $y=-0.01344$, $p_x=0.49982$, $p_y=0$



Applications – Hénon-Heiles system



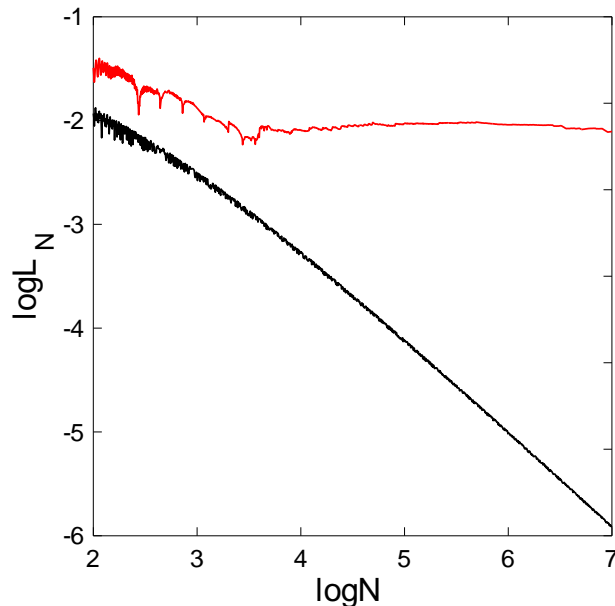
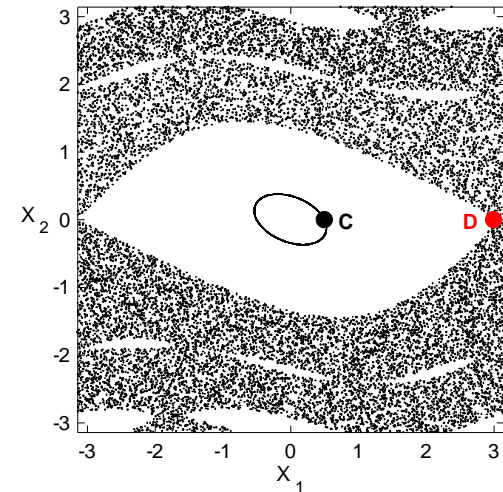
Applications – 4D map

$$\begin{aligned}
 \mathbf{x}'_1 &= \mathbf{x}_1 + \mathbf{x}_2 \\
 \mathbf{x}'_2 &= \mathbf{x}_2 - \nu \sin(\mathbf{x}_1 + \mathbf{x}_2) - \mu [1 - \cos(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4)] \\
 \mathbf{x}'_3 &= \mathbf{x}_3 + \mathbf{x}_4 \\
 \mathbf{x}'_4 &= \mathbf{x}_4 - \kappa \sin(\mathbf{x}_3 + \mathbf{x}_4) - \mu [1 - \cos(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4)]
 \end{aligned} \pmod{2\pi}$$

For $\nu=0.5$, $\kappa=0.1$, $\mu=0.1$ we consider the orbits:

regular orbit C with initial conditions $x_1=0.5$, $x_2=0$, $x_3=0.5$, $x_4=0$.

chaotic orbit D with initial conditions $x_1=3$, $x_2=0$, $x_3=0.5$, $x_4=0$.



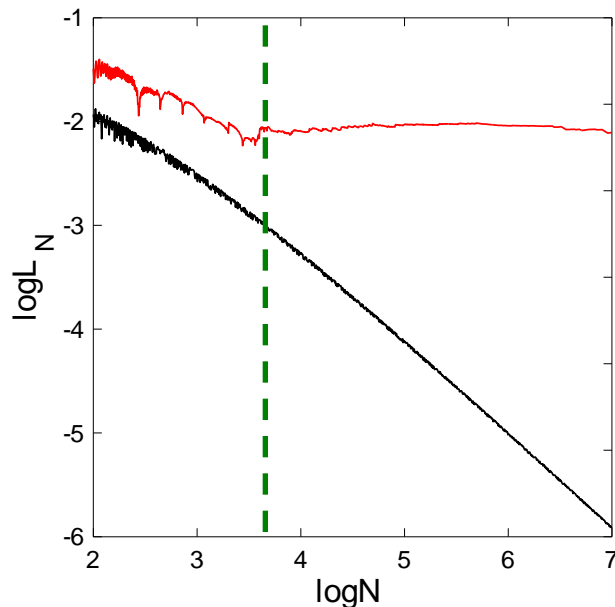
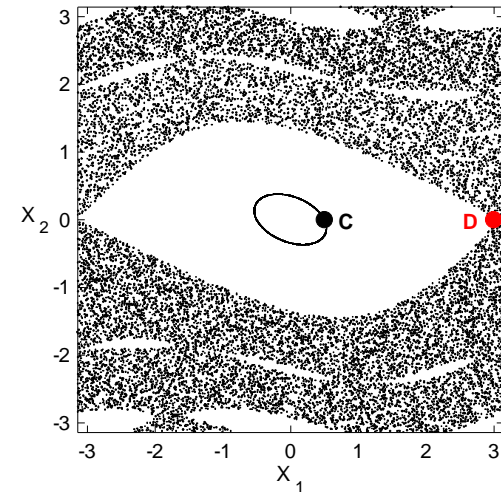
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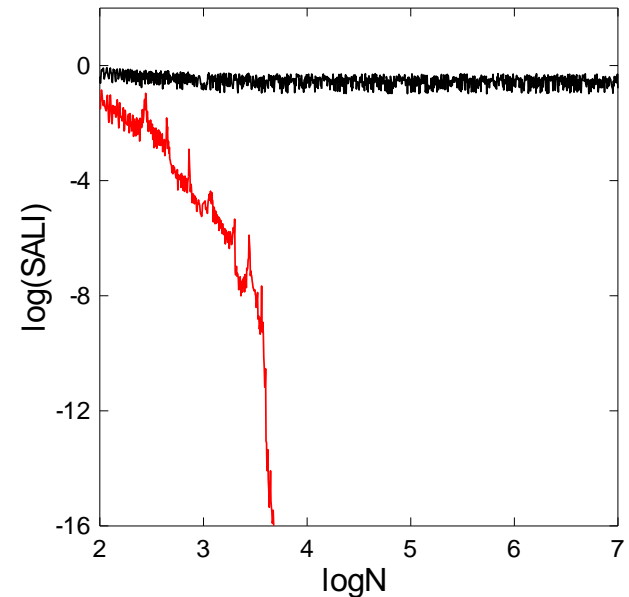
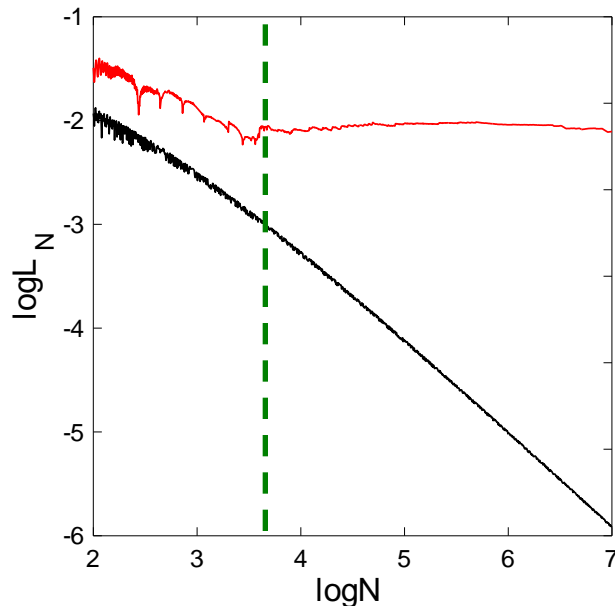
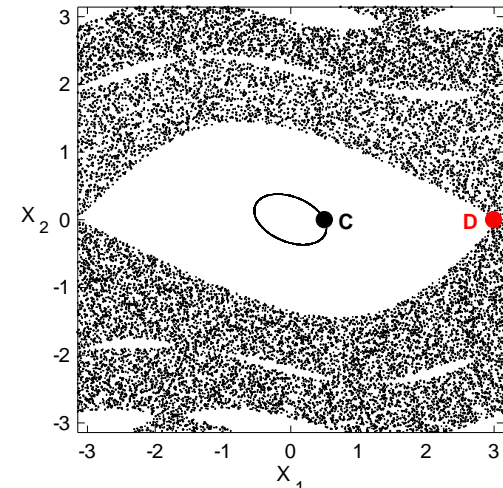
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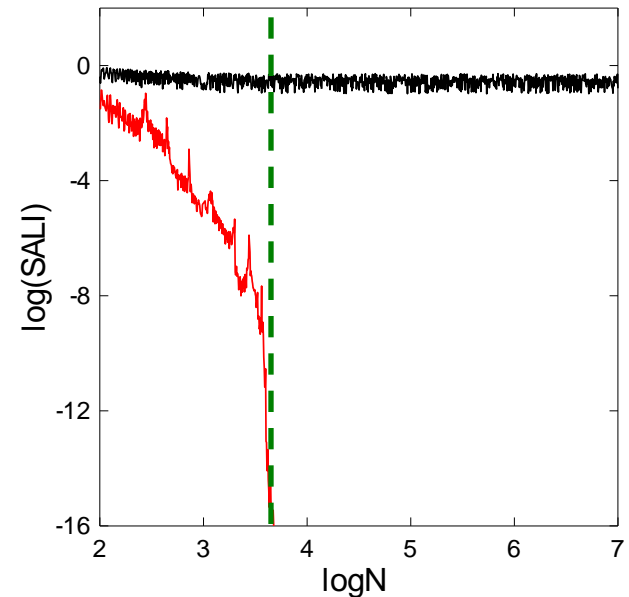
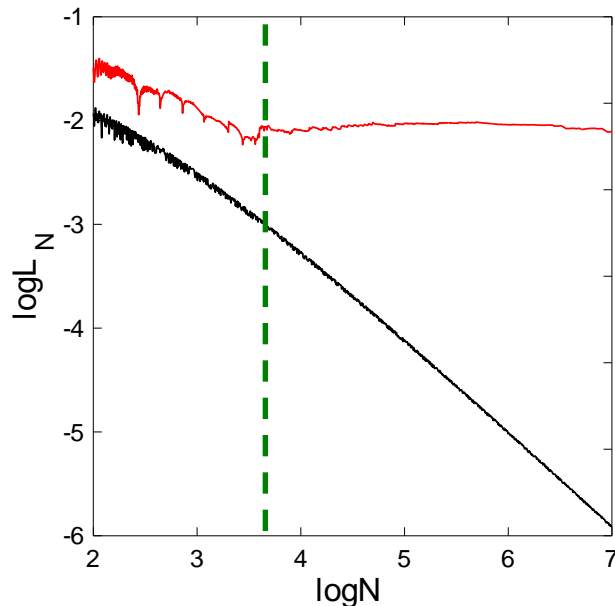
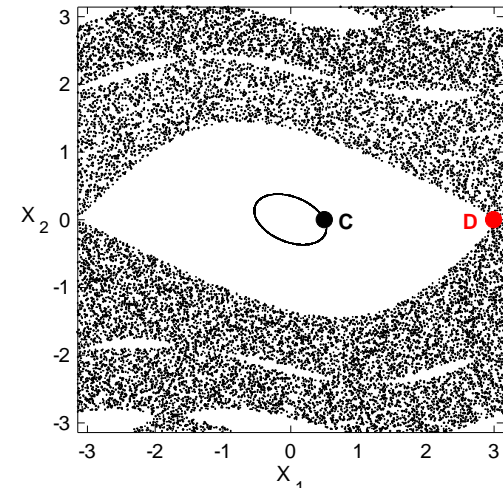
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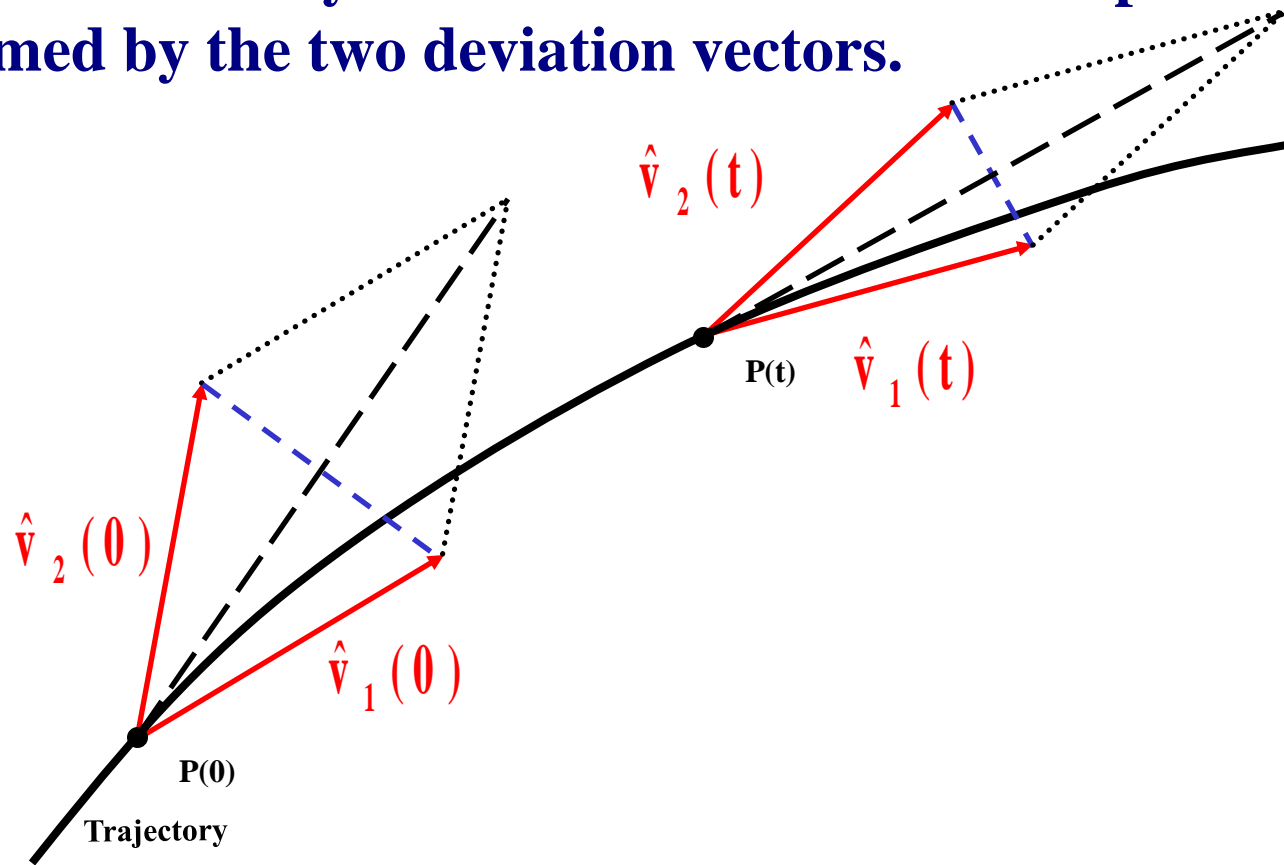
chaotic orbit D with initial conditions $x_1=3$, $x_2=0$, $x_3=0.5$, $x_4=0$.



The Generalized ALignment Indices (GALIs) method

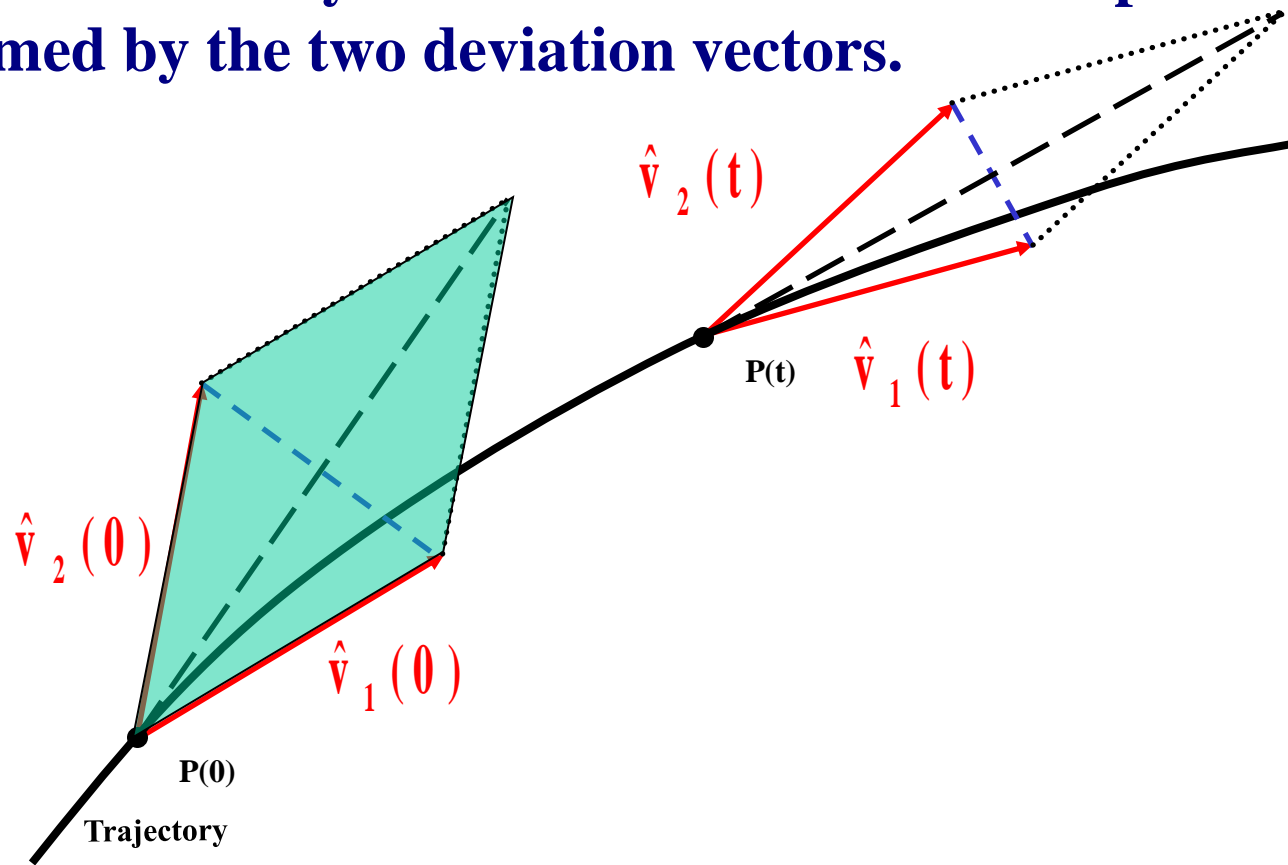
Definition of the Generalized Alignment Index (GALI)

SALI effectively measures the 'area' of the parallelogram formed by the two deviation vectors.



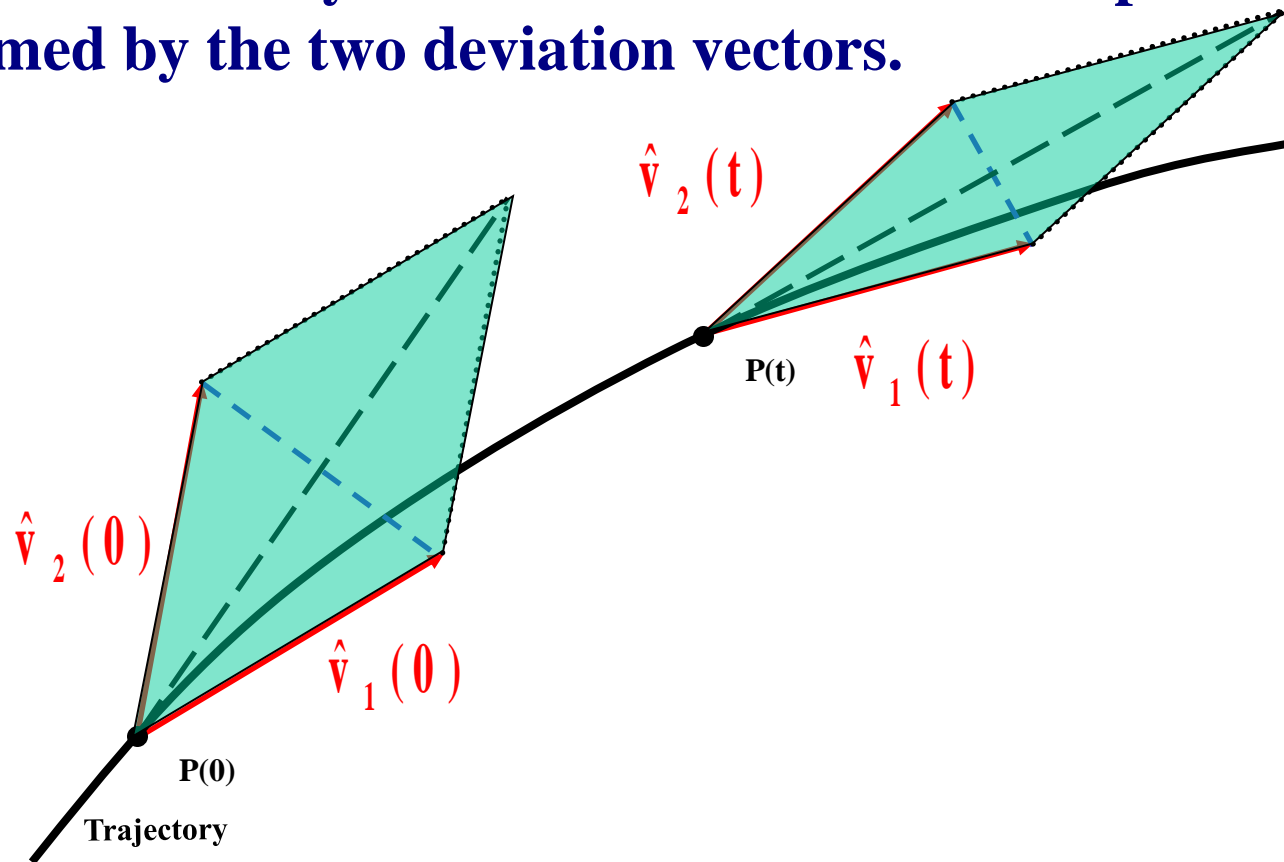
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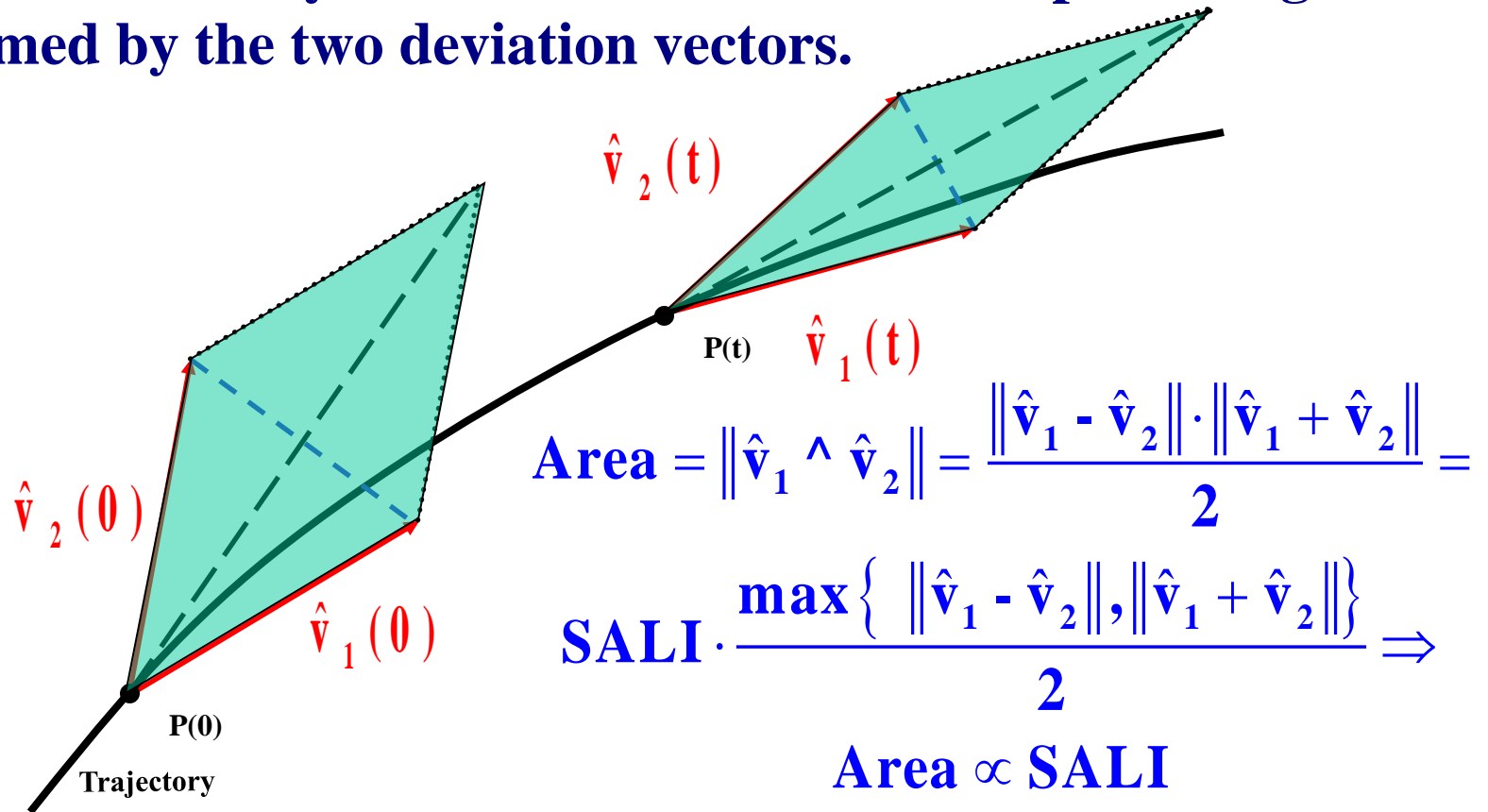
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Definition of the GALI

In the case of an N degree of freedom Hamiltonian system or a $2N$ symplectic map we follow the evolution of

k deviation vectors with $2 \leq k \leq 2N$,

and define (Ch.S., Bountis, Antonopoulos, 2007, Physica D) the Generalized Alignment Index (GALI) of order k :

$$GALI_k(t) = \left\| \hat{v}_1(t) \wedge \hat{v}_2(t) \wedge \dots \wedge \hat{v}_k(t) \right\|$$

where

$$\hat{v}_1(t) = \frac{\mathbf{v}_1(t)}{\|\mathbf{v}_1(t)\|}$$

Behavior of the $GALI_k$ for chaotic motion

$GALI_k$ ($2 \leq k \leq 2N$) tends exponentially to zero with exponents that involve the values of the first k largest Lyapunov exponents $\sigma_1, \sigma_2, \dots, \sigma_k$:

$$GALI_k(t) \propto e^{-[(\sigma_1 - \sigma_2) + (\sigma_1 - \sigma_3) + \dots + (\sigma_1 - \sigma_k)]t}$$

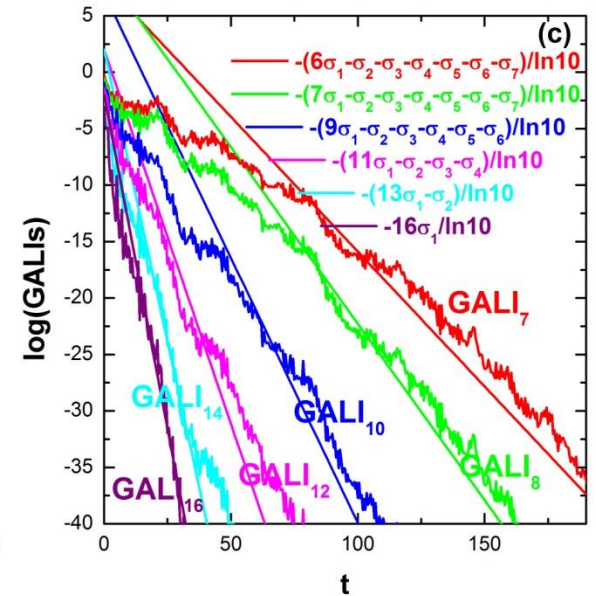
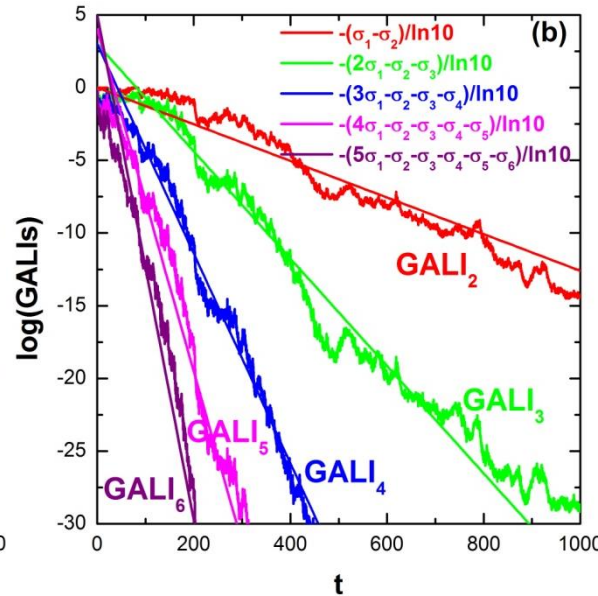
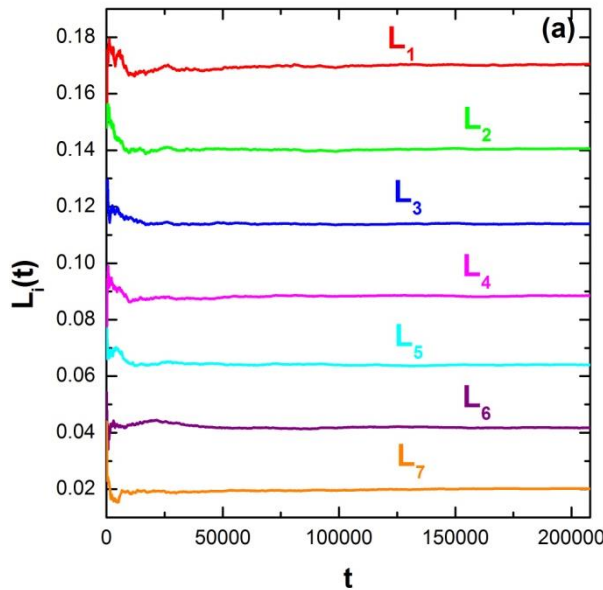
The above relation is valid even if some Lyapunov exponents are equal, or very close to each other.

Behavior of the $GALI_k$ for chaotic motion

N particles Fermi-Pasta-Ulam (FPU) system:

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 + \sum_{i=0}^N \left[\frac{1}{2} (q_{i+1} - q_i)^2 + \frac{\beta}{4} (q_{i+1} - q_i)^4 \right]$$

with fixed boundary conditions, $N=8$ and $\beta=1.5$.



Behavior of the GALI_k for regular motion

If the motion occurs on an **s-dimensional torus** with $s \leq N$ then the behavior of GALI_k is given by (Ch.S., Bountis, Antonopoulos, 2008, Eur. Phys. J. Sp. Top.):

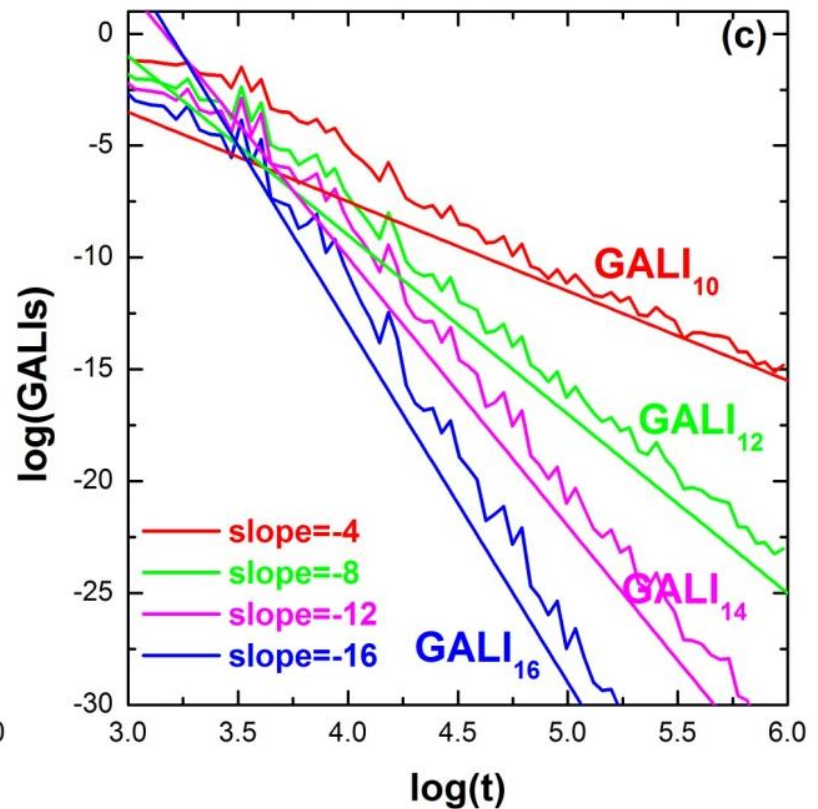
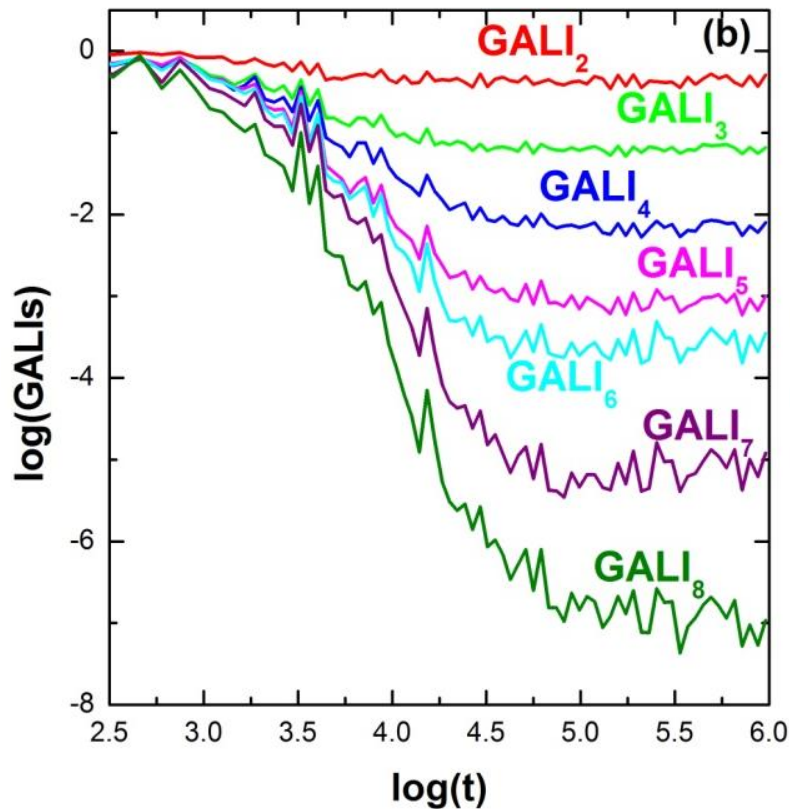
$$\text{GALI}_k(t) \propto \begin{cases} \text{constant} & \text{if } 2 \leq k \leq s \\ \frac{1}{t^{k-s}} & \text{if } s < k \leq 2N - s \\ \frac{1}{t^{2(k-N)}} & \text{if } 2N - s < k \leq 2N \end{cases}$$

while in the **common case with $s=N$** we have :

$$\text{GALI}_k(t) \propto \begin{cases} \text{constant} & \text{if } 2 \leq k \leq N \\ \frac{1}{t^{2(k-N)}} & \text{if } N < k \leq 2N \end{cases}$$

Behavior of the $GALI_k$ for regular motion

N=8 FPU system



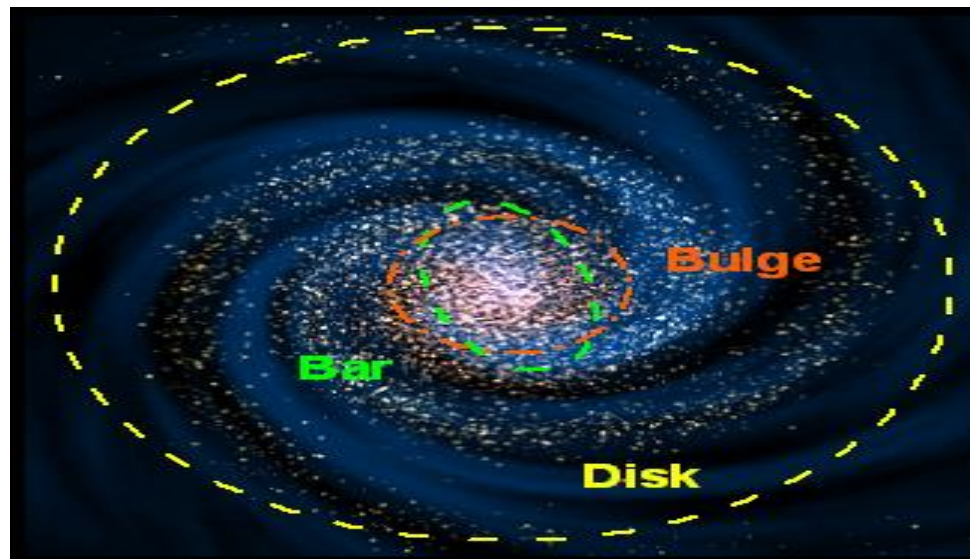
A time-dependent Hamiltonian system

Barred galaxies

NGC 1433



NGC 2217



Barred galaxy model

The 3D bar rotates around its short z -axis (x : long axis and y : intermediate). The Hamiltonian that describes the motion for this model is:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z) - \Omega_b(xp_y - yp_x) \equiv \text{Energy}$$

This model consists of the superposition of potentials describing an **axisymmetric** part and a **bar** component of the galaxy (**Manos, Bountis, Ch.S., 2013, J. Phys. A**).

a) Axisymmetric component:

i) **Plummer sphere:**

$$V_{\text{sphere}}(x, y, z) = -\frac{GM_s}{\sqrt{x^2 + y^2 + z^2 + \epsilon_s^2}}$$

ii) **Miyamoto–Nagai disc:**

$$V_{\text{disc}}(x, y, z) = -\frac{GM_D}{\sqrt{x^2 + y^2 + (A + \sqrt{B^2 + z^2})^2}}$$

b) Bar component: $V_{\text{bar}}(x, y, z) = -\pi Gabc \frac{\rho_c}{n+1} \int_{\lambda}^{\infty} \frac{du}{\Delta(u)} (1 - m^2(u))^{n+1},$

(Ferrers bar)

$$\rho_c = \frac{105}{32\pi} \frac{GM_B}{abc}$$

$$\text{where } m^2(u) = \frac{x^2}{a^2 + u} + \frac{y^2}{b^2 + u} + \frac{z^2}{c^2 + u}, \Delta^2(u) = (a^2 + u)(b^2 + u)(c^2 + u),$$

n : positive integer ($n = 2$ for our model), λ : the unique positive solution of $m^2(\lambda) = 1$

Its density is:

$$\rho = \begin{cases} \rho_c (1 - m^2)^n, & \text{for } m \leq 1 \\ 0, & \text{for } m > 1 \end{cases}, \text{ where } m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, a > b > c \text{ and } n = 2.$$

Time-dependent barred galaxy model

The 3D bar rotates around its short z -axis (x : long axis and y : intermediate). The Hamiltonian that describes the motion for this model is:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z, t) - \Omega_b(xp_y - yp_x) \equiv \text{Energy}$$

This model consists of the superposition of potentials describing an **axisymmetric** part and a **bar** component of the galaxy (Manos, Bountis, Ch.S., 2013, J. Phys. A).

a) Axisymmetric component:

$$M_S + M_B(t) + M_D(t) = 1, \text{ with } M_B(t) = M_B(0) + \alpha t$$

i) **Plummer sphere:**

$$V_{\text{sphere}}(x, y, z) = -\frac{GM_S}{\sqrt{x^2 + y^2 + z^2 + \epsilon_s^2}}$$

ii) **Miyamoto–Nagai disc:**

$$V_{\text{disc}}(x, y, z) = -\frac{GM_D(t)}{\sqrt{x^2 + y^2 + (A + \sqrt{B^2 + z^2})^2}}$$

b) Bar component: $V_{\text{bar}}(x, y, z) = -\pi Gabc \frac{\rho_c}{n+1} \int_{\lambda}^{\infty} \frac{du}{\Delta(u)} (1 - m^2(u))^{n+1},$

(Ferrers bar)

$$\text{where } m^2(u) = \frac{x^2}{a^2 + u} + \frac{y^2}{b^2 + u} + \frac{z^2}{c^2 + u}, \Delta^2(u) = (a^2 + u)(b^2 + u)(c^2 + u),$$

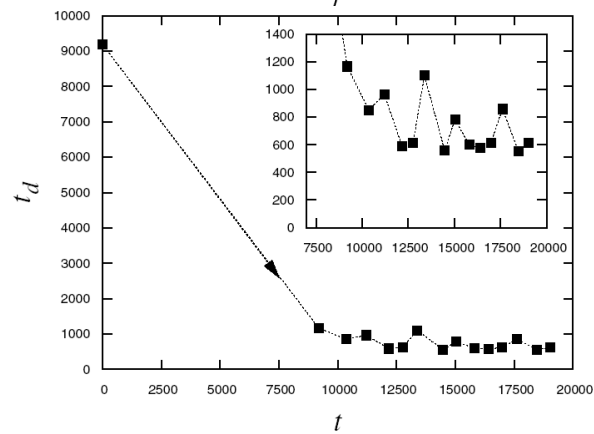
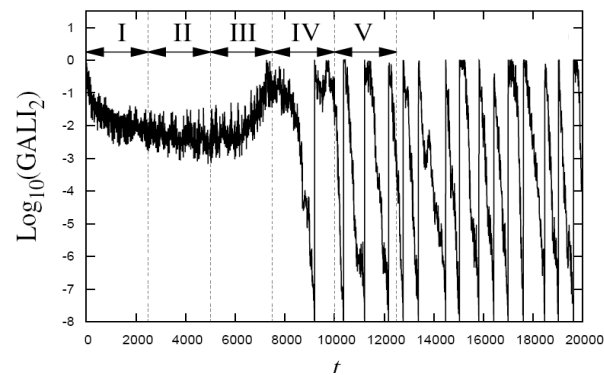
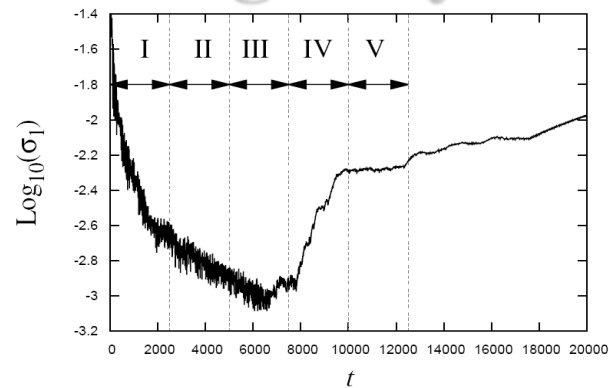
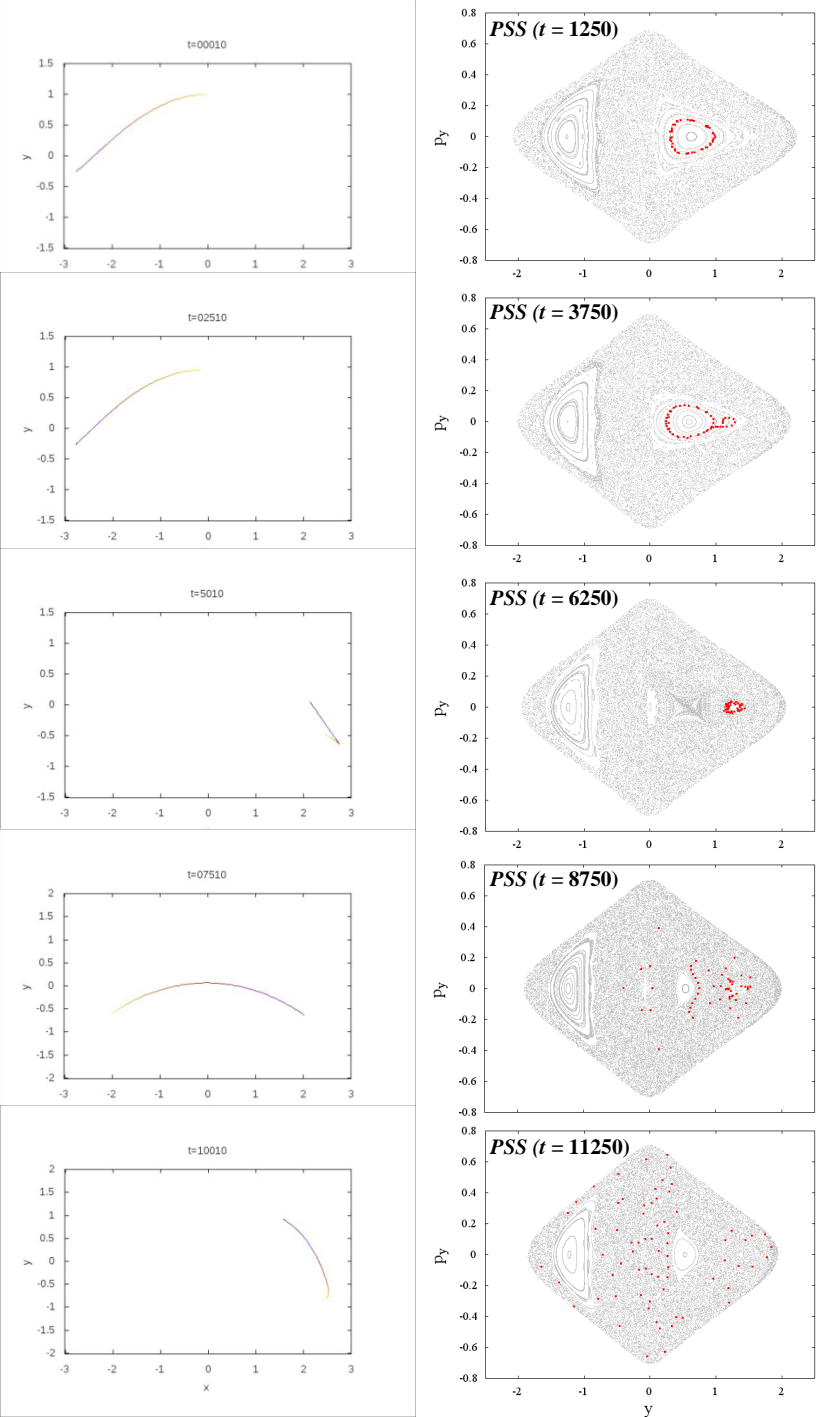
n : positive integer ($n = 2$ for our model), λ : the unique positive solution of $m^2(\lambda) = 1$

$$\rho_c = \frac{105}{32\pi} \frac{GM_B(t)}{abc}$$

Its density is:

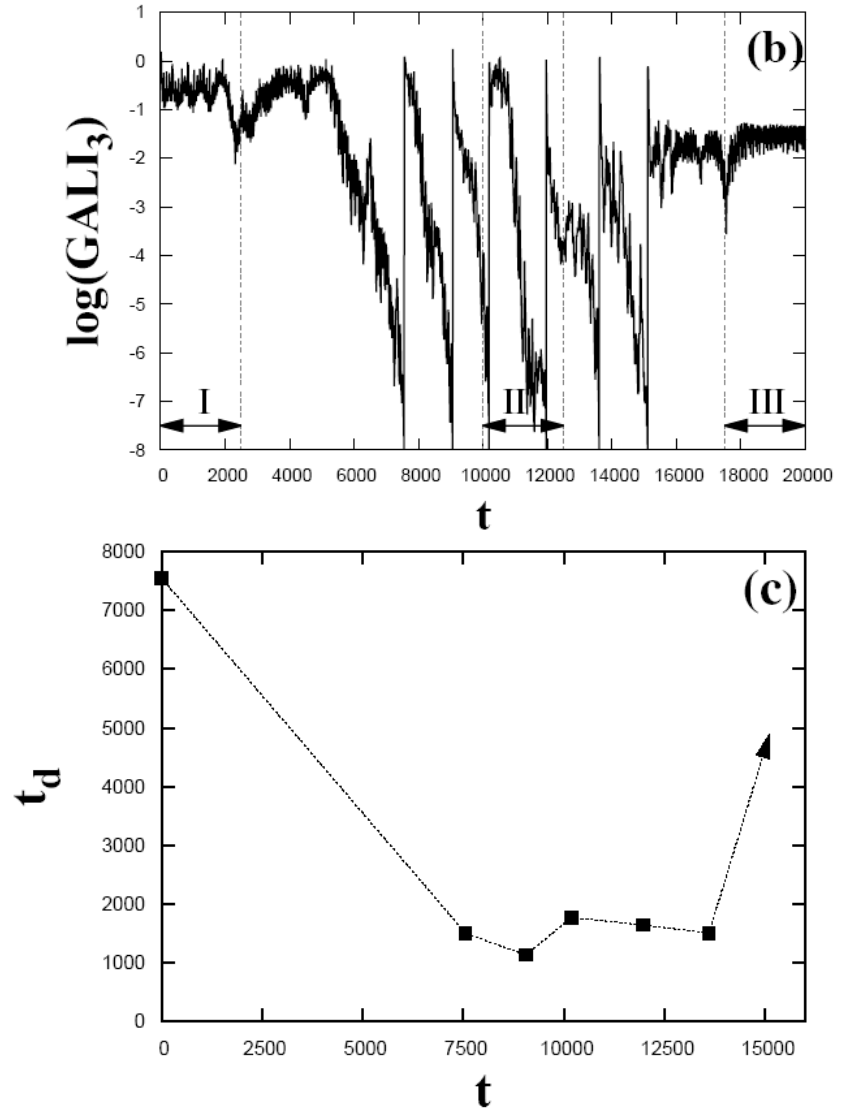
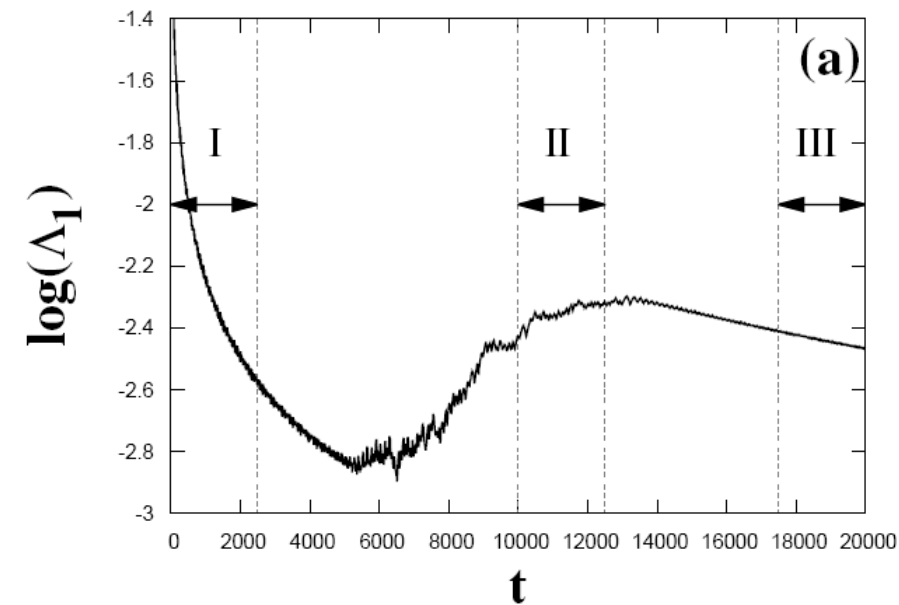
$$\rho = \begin{cases} \rho_c (1 - m^2)^n, & \text{for } m \leq 1 \\ 0, & \text{for } m > 1 \end{cases}, \text{ where } m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, a > b > c \text{ and } n = 2.$$

Time-dependent 2D barred galaxy model



Time-dependent 3D barred galaxy model

Interplay between chaotic and regular motion



Summary

- We discussed methods of chaos detection based on
 - ✓ the visualization of orbits
 - ✓ the numerical analysis of orbits
 - ✓ the evolution of deviation vectors (variational equations – tangent map)
- The Smaller (SALI) and the Generalized (GALI) ALignment Index methods are **fast, efficient and easy to compute chaos indicator**.
- Behaviour of the Generalized ALignment Index of order k ($GALI_k$):
 - ✓ **Chaotic motion: it tends exponentially to zero**
 - ✓ **Regular motion: it fluctuates around non-zero values** (or goes to zero following power-laws)
- $GALI_k$ indices :
 - ✓ can **distinguish rapidly and with certainty between regular and chaotic motion**
 - ✓ can be used to characterize **individual orbits as well as "chart" chaotic and regular domains** in phase space
 - ✓ can identify regular **motion on low-dimensional tori**
 - ✓ are perfectly suited for **studying the global dynamics of multidimensional systems, as well as of time-dependent models**

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